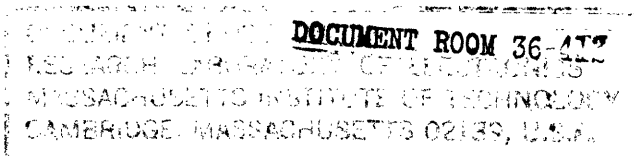


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NOISE SOURCES DESCRIBING QUANTUM EFFECTS IN THE LASER OSCILLATOR

HERMAN J. PAUWELS

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LASER OSCILLATOR

Herman J. Pauwels

This report is based on a thesis submitted to the Department of Electrical Engineering, M.I.T., January 17, 1966, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Abstract

With quantum noise in the laser we mean those properties of the laser output that are caused by the quantum nature of the electromagnetic field and of the material systems and reservoirs with which it interacts. It is shown that a fully quantum-mechanical treatment of the laser can be formulated in a noise-source formalism, for which the noise sources are operators. A definition is given for the Gaussian character of operators in an appropriate ensemble, and it is shown that the noise sources for the laser are Gaussian. Special laser models are treated. In the first model we require that the relevant relaxation time constants of the material be much smaller than those of the field; in the second we drop this restriction. The final operator equations are solved by means of a linearization approximation that is only justified for operation points "sufficiently" above threshold. In the first model we take the quantum nature of the field above threshold (or equivalently the commutator of certain noise-source operators) consistently into account; in the second model we neglect these quantum effects. The results are compared with the predictions of a "semiclassical" theory in which classical equations contain noise sources that correctly represent properties of the field below threshold.

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I. INTRODUCTION

This report will deal with quantum noise in the laser oscillator. We want to show that a fully quantum-mechanical treatment of the laser can be formulated in a noise-source formalism with the noise sources as operators. We shall show that the moments of these noise sources have Gaussian properties. We then want to obtain solutions for various moments of the variables of the electromagnetic field, in operation points both below and above threshold.

1.1 DEFINITION OF QUANTUM NOISE

With quantum noise in the laser we mean that we are dealing with those properties of the output of the laser that are caused by the quantum nature of the electromagnetic field and of the material system and the reservoirs with which it interacts. It is necessary to give a fully quantized treatment of the laser. This report differs from most other theories of lasers, which treat the electromagnetic field classically. With these theories one uses a self-consistent approach in which the electromagnetic field enters as a c-number variable into the quantum-mechanical material equations, and the material enters into the classical Maxwell's equations as a c-number polarization. This polarization is the expectation value of the polarization operator of the material. In this procedure one loses the quantum noise caused by the quantum nature of both the electromagnetic field and the material. These theories are thus clearly unacceptable for our purposes.

The output of the laser field is characterized by various moments of the field variables. The most important moments in which we are interested are the first-order Glauber function G_1 , the second-order Glauber function G_2 , the expectation value of the field commutator, and the correlation function of the photon number. If we consider single-mode operation of the laser oscillator, and if we call a^+ and a the creation and annihilation operators of that mode, then we define these functions as

$$G_1(\tau) = \langle a^+(t+\tau)a(\tau) \rangle \quad (1)$$

$$G_2(\tau) = \langle T^+[a^+(t)a^+(t+\tau)]T[a(t+\tau)a(t)] \rangle \quad (2)$$

$$\langle [a(t+\tau), a^+(t)] \rangle \quad (3)$$

$$G_2^!(\tau) = \langle n(t+\tau)n(t) \rangle, \quad (4)$$

where T is a time-ordering operator that puts the later time first, T^+ is a time-ordering operator that puts the earlier time first, and n is the photon number operator a^+a . Note that

$$G_1(0) = \langle n(t) \rangle \quad (5)$$

$$G_2(0) = G_2'(0) - G_1(0). \quad (6)$$

Physical interpretation in quantum mechanics deals with the interpretation of moments (of Hermitian operators). It is always possible to give to a specific moment either a classical field or a classical particle interpretation. It is not always possible to give a single classical interpretation to several moments. We shall show, for instance, that $G_1(\tau)$ and $G_2(\tau)$ in the steady state below threshold can be interpreted by means of a classical field in a high-Q cavity driven by Gaussian noise sources, the second-order moments of which account for spontaneous emission and Nyquist noise. We shall show that $G_1(0)$ and $G_2'(\tau)$ below threshold can be best interpreted by means of a classical-particle theory in which the photon number is linearly attenuated and is driven by spontaneous emission and thermal excitations and restricted to integer values. This restriction gives rise to shot noise in $G_2'(\tau)$. Above threshold one can neglect the spontaneous emission and thermal drives, and the most important remaining source of noise in $G_2'(\tau)$ is the shot noise associated with the restriction of n (and also of the inversion of the material) to integer values.

These remarks make it clear why we prefer to use the general name "quantum noise" for those noise properties of the laser output which are caused by the quantum nature of the electromagnetic field and the material. Only when we consider specific moments and operation points, shall we interpret the noise as spontaneous-emission noise or shot noise, or as combination of both.

1.2 ARGUMENTS FOR OPERATOR NOISE-SOURCE FORMALISM

Until recently, the quantized treatment of the laser has been restricted to the laser amplifier and the laser oscillator below threshold. These are linear devices and their quantum description leads to a set of linear operator equations that can relatively easily be solved. From the solutions one can derive, for given initial conditions, various moments of the field and of the photon number, at a single time¹ or at different times.

These results have been rederived from "equivalent classical models."² By this one means a set of classical equations that leads to the same moments as the quantum treatment. The justification for the equivalent model pre-supposes the knowledge of the results of the quantum treatment. It appears remarkable that two entirely different calculations lead to the same result. We therefore suspected that it would be possible to reformulate the quantum equations in such a way that they would resemble formally the equations of the equivalent classical model and that the method for solving the former would be in step-by-step analogy with the method for solving the latter. The part played by the equivalent model would then be to suggest the technique for solving the quantum problem.

For the linear laser device the equivalent model leads to the solution of a set of linear differential equations for the field variables driven by noise sources. We therefore suspect that a quantum analogue to a classical noise source should exist. We shall call

it an "operator noise source." Furthermore, the classical noise sources are Gaussian processes. We expect that the operator noise source would be a "Gaussian operator process."

For the linear laser device the moments of the field variables were originally derived through various methods by several authors¹ and the operator noise-source approach would therefore only add a new method for deriving the field moments to the many already existing methods. For the laser oscillator above threshold, which is a nonlinear device, no quantum solution existed. Haus³ has considered a simple laser model and has derived the semiclassical field equation for this model (Van der Pol equation). He inserted the noise source of the equivalent linear problem into this equation as a driving source. He supposed that this driven equation was an equivalent classical model of the quantum model. It was impossible to prove that it really was equivalent because no quantum solution existed. He solved this equation for the amplitude and phase fluctuations of the laser field. Experiments⁴ have confirmed the correctness of the results for the amplitude fluctuations within approximately 10%. This suggests that it should be possible to give a quantum formulation that is a step-by-step analogue of the "equivalent" model. This formulation and solution would then constitute a theoretical proof of the results of Haus.³ It is the purpose of this report to show that such a formulation exists and to solve it for the phase and amplitude fluctuations or, expressed more correctly, for the first-order and second-order Glauber functions.⁵ We shall then try to obtain noise-source formulations for more general models than the one considered by Haus.³

1.3 OUTLINE

In Section II we describe the variables and Hamiltonians of the four systems that are essential in any laser model, the field system and its reservoir (loss reservoir), and the material system and its reservoir. We also describe the coupling between these systems.

In Section III, we show that the concept of operator noise sources exists for the linear laser device, and, by means of this noise-source formulation, results can be obtained for second-order moments of the field variables in the steady state of the laser oscillator below threshold.

In Section IV, we introduce and discuss the concept of operators that are Gaussian in an appropriate ensemble. We apply these concepts to obtain results for higher order moments of the field variables in the steady state of the laser oscillator below threshold. We give physical interpretations to some of these results.

In Section V we give a noise-source formulation for the laser oscillator above threshold. We restrict ourselves to the laser model of the original theory of Haus.³ Our results contain corrections to Haus' results which are too small to have been detected experimentally.⁴

In Section VI we give a sufficiently general noise-source formulation of the interaction of a many-level system with appropriate reservoirs. We apply this formulation to a

more general model for the laser oscillator above threshold. We solve this formulation by means of an approximation method that neglects certain "quantum" corrections. This procedure is qualitatively justified through the results of Section V. We give physical interpretations.

In Section VII we discuss the relationship between moments of the field in the laser cavity and moments of the field in the laser beam. The proof of these results will be found in the thesis on which this report is based.

In Section VIII we summarize and interpret our main results. We compare our methods and results with those of other authors.

In Appendix A we explain the method by which we obtained correlation functions in the steady state of a set of linear (operator) equations. In Sections V and VI we use a linearization approximation. Although the quantum-mechanical consistency of this approximation has not yet been proved in a fully satisfactory way, one can obtain an estimate of the "linearization errors" by solving the "semiclassical" equations of the original theory of Haus³ by means of the Volterra-kernel method for nonlinear-feedback systems driven by Gaussian noise sources. The results of this method are summarized. They show that the linearization approximation is completely unacceptable in the immediate neighborhood of threshold but becomes rapidly very good somewhat above threshold.

In Appendix C we show under which conditions and how one can transform a special type of operator equations to the Langevin form. The techniques developed in this Appendix will be used throughout this report; they form an essential part of the noise-source formulation given in section 6.3. Finally, in Appendix D we analyze a loaded LC circuit. We often refer to such a circuit throughout this report.

1.4 RELATED WORK

Glauber⁵ and Kelley and Kleiner⁶ explained the importance of Glauber functions in photocounting experiments. They do not attempt to calculate Glauber functions for specific sources of radiation. We are interested in a specific source, i. e., the laser oscillator below and above threshold.

We first mention the relevant work on noise in laser models, in which the relaxation times of the material are much smaller than the cavity relaxation time. A quantum analysis of the laser amplifier and laser oscillator below threshold is given by Louisell.¹ Equivalent models for these linear devices have been discussed by Haus and Mullen.² These models are extended by Haus to the laser oscillator above threshold; however, without proof of the "equivalence" of the model.³ His theory has been the basis for the interpretation of photon counting experiments on the He-Ne laser, performed by Freed and Haus.^{4, 7} They discuss extensively the behavior of intensity fluctuations (or second-order Glauber function) as a function of operation points; we shall not repeat these discussions. As we have mentioned, these results are rederived, proved, and eventually corrected by means of the operator noise-source formulation in Sections III, IV, and V.

Shimoda, Takahasi, and Townes discussed an equivalent model for intensity

fluctuations in the laser below threshold, based on rate equations and shot noise.⁸ McCumber extended this model to the laser oscillator above threshold; however, without proof of the equivalence of his model.⁹ His results contain Haus' results for the intensity fluctuations, but describe also the effect of finite relaxation time of the inversion. The validity of rate equations requires the assumption of very large material linewidth. He discusses extensively the behavior of intensity fluctuations as a function of operation points and relaxation times. These discussions will not be repeated in this report.

The operator noise-source formulation of a laser model with no restrictions on the various relaxation times, requires first an operator noise-source formulation of the interaction between a many-level system and appropriate reservoirs. Such a formulation is given by Lax,¹⁰ and Haken and Weidlich.¹¹ They essentially assume that the second-order moments of the noise sources are proportional to Dirac functions. By choosing specific models for the reservoirs, more exact but less general derivations are given by Sauermann,¹² and ourselves (in Section VI of this report). We believe that our derivation is more general than Sauermann's. We also discuss phenomenological models.

Lax applies his results to obtain the phase fluctuations (first-order Glauber function) of the laser oscillator.¹³ A joint publication with Lax on intensity fluctuations has been announced by McCumber.⁹ Haken applied his formalism to obtain results for intensity and phase fluctuations.¹⁴ Our results in Section VI agree with Haken's results. These results prove, among other things, the "equivalence" of McCumber's model.⁹ Earlier results of Haken are not completely correct.¹⁵

Other methods from the operator noise-source formulation are used by Korenmann,¹⁶ and Scully, Lamb, and Stephen.¹⁷ In Section VIII we briefly discuss their methods. They obtained results for the phase fluctuations.

The noise-source formulations constitute a linearization approximation that is only justified sufficiently above threshold. An approach based on the Fokker-Planck equation and that can be used through the threshold region has been discussed, in a semiclassical context, by Hempsteadt and Lax,¹⁸ and by Risken.¹⁹ Schmid and Risken have announced a forthcoming paper on the quantum analysis based on the Fokker-Planck equation.²⁰ These approaches will not be discussed in this report.

II. QUANTUM FORMALISM

We shall now describe the quantum formalism for the various interacting systems that constitute a laser. The Hamiltonian and the Heisenberg equations of motion derived here will be used throughout this report.

2.1 INTERACTING SYSTEMS

The four systems that are essential to any laser model are the field system, the material system, the field reservoir (loss system), and the material reservoir. Throughout this report we shall restrict ourselves to a field system consisting of one mode of the (closed) laser cavity. Its resonance angular frequency will be denoted ω_0 . The material system consists of a large number, N , of many-level systems. We shall call each many-level system a particle. It consists, in fact, of one electron in a molecule or ion. We distinguish the various particles by the index j . We suppose that in each particle j there is one level pair with a resonance angular frequency, ω_j , either equal or close to ω_0 . These two levels are dipole coupled to the field mode. The field reservoir (loss system) consists of a set of harmonic oscillators that are coupled to the field mode, and are initially in thermal equilibrium. The material reservoir is coupled to the material system. It causes pumping, nonradiative decay, and randomization of the material system.

We shall assume weak coupling among the various systems. This allows us to write all interaction Hamiltonians in the rotating-wave approximation, that is, double-frequency terms are neglected. We shall work in the Heisenberg formalism in which the equation of motion of an arbitrary operator O is given by

$$\frac{dO}{dt} = \frac{1}{i\hbar} [O, H], \quad (7)$$

where H is the Hamiltonian. We shall often eliminate the natural time dependence of operators, usually without changing the notation for the new slowly time-variant operators.

2.2 FORMALISM FOR THE ELECTROMAGNETIC FIELD

If the field mode is coupled to no other system, its space-time dependence is governed by Maxwell's equations

$$\text{curl } E = -\mu \frac{\partial H}{\partial t}; \text{curl } H = \epsilon \frac{\partial E}{\partial t}. \quad (8)$$

We set

$$E = (\hbar\omega_0/2\epsilon)^{1/2} e(r) i[a(t)-a^\dagger(t)] \quad (9a)$$

$$H = (\hbar\omega_0/2\mu)^{1/2} h(r) [a(t)+a^\dagger(t)], \quad (9b)$$

where the mode functions $e(r)$ and $h(r)$ obey

$$\text{curl } e(r) = \omega_0 \sqrt{\epsilon \mu} h(r); \text{div } e(r) = 0; \int_{\text{cav.}} e(r) \cdot e(r) dr = 1$$

$$\text{curl } h(r) = \omega_0 \sqrt{\epsilon \mu} e(r); \text{div } h(r) = 0; \int_{\text{cav.}} h(r) \cdot h(r) dr = 1.$$

If we substitute Eqs. 9 in Eqs. 8, we obtain

$$\frac{da(t)}{dt} = -i\omega_0 a(t); \frac{da^\dagger(t)}{dt} = i\omega_0 a^\dagger(t).$$

These equations can be derived from the Hamiltonian H_f

$$H_f = \hbar\omega_0 a^\dagger a = \int_{\text{cav.}} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dr - \frac{1}{2} \hbar\omega_0, \quad (10)$$

where the operators a and a^\dagger are Hermitian conjugate (h. c.) operators that obey the commutation relations

$$[a, a^\dagger] = 1; [a, a] = [a^\dagger, a^\dagger] = 0. \quad (11)$$

The operators a^\dagger , a , and $a^\dagger a = n$ are the creation, annihilation, and photon number operators of the cavity mode. For other properties we refer to Louisell's work.¹

2.3 FORMALISM FOR A MANY-LEVEL SYSTEM

We shall now use the second quantization formalism for a single system as developed by Lax.²¹ If a_i^\dagger and a_i create and annihilate the system in the energy eigenstate $|i\rangle$, then the properties of the operators

$$s_{ij} = a_i^\dagger a_j = |i\rangle\langle j| \quad (12)$$

can be summarized as

$$s_{ij}s_{kl} \cdots s_{mn}s_{pq} = s_{iq}\delta_{jk}\delta_{l-} \cdots \delta_{-m}\delta_{np}. \quad (13)$$

$$\sum_i s_{ii} = 1 \quad (14)$$

If i is different from j , then s_{ij} is the transition operator from level j into level i ; if i equals j , then s_{ii} is the population operator of level i . If we denote by $\hbar\Omega_i$ the energy of level i , then the Hamiltonian of the many-level system is given by

$$H = \sum_i \hbar\Omega_i s_{ii} = \hbar\Omega_i s_{ii} \quad (15)$$

and when the system is not coupled to any other system, the equation of motion of s_{ij} is

$$\frac{ds_{ij}(t)}{dt} = i\omega_{ij}s_{ij}. \quad (16)$$

Here we have set $\omega_{ij} = \Omega_i - \Omega_j$, with $\omega_{ij} > 0$ when level i is above level j ; with $\omega_{ij} < 0$ when i is below j ; and with $\omega_{ij} = 0$ when $i = j$.

If $-q$ is the electronic charge, and r the position operator of the electron in the many-level system, then the polarization operator is given by

$$P = \sum_{i,j} |i\rangle \langle i| -qr |j\rangle \langle j| = \mu_{ij}s_{ij}$$

$$\mu_{ij} = \langle i| -qr |j\rangle.$$

We have said that the material system consists of a large number of such many-level systems (particles) and that in each particle (j) there is a level pair that has a resonance angular frequency ω_j close to ω_0 . For this level pair of particle j we shall use the following notation:

$$p_j^- = (s_{12})_j; p_j^+ = (s_{21})_j; p_j = (s_{22} - s_{11})_j; q_j = (s_{22} + s_{11})_j, \quad (17)$$

where $|2\rangle$ or $|+\rangle$ is the upper level and $|1\rangle$ or $|-\rangle$ the lower level (Fig. 1). The operator p_j^+ lifts the electron from the lower level to the upper level; p_j^- does the opposite.

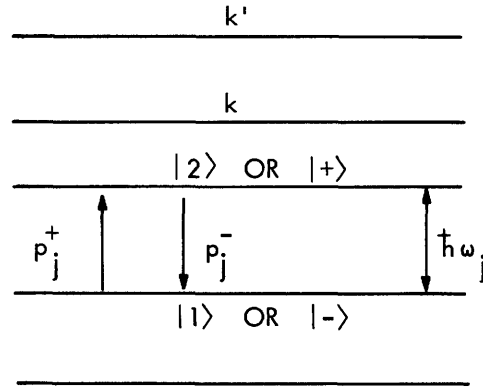


Fig. 1. Many-level system j .

The operator p_j is the population-difference operator; its expectation value is positive if the population of the level pair is inverted. The operator q_j is the population-sum operator; it is the identity operator for a strictly two-level system. We can always choose the phase of the states $|1\rangle$ and $|2\rangle$ such that

$$\mu_j = \langle 1| -qr |2\rangle = \langle 2| -qr |1\rangle = \text{real}, \quad (18)$$

so that the polarization operator for this level pair is

$$P_j = \mu_j (p_j^+ + p_j^-). \quad (19)$$

Since Eq. 16 shows that p_j^+ has a positive frequency dependence, we shall also say that p_j^+ is proportional to the positive-frequency component of the polarization operator; p_j^- is proportional to the negative-frequency component of the polarization operator.

We assume that the coupling of these particular level pairs of the various particles to the field mode is described by the interaction Hamiltonian

$$H_{fm} = i\hbar \sum_j \kappa_j (a^+ p_j^- - p_j^+ a), \quad (20)$$

where the coupling constant κ_j is given by

$$\kappa_j = \frac{1}{\hbar} (\hbar\omega_0/2\epsilon)^{1/2} \mu_j \cdot e(r_j), \quad (21)$$

and r_j is the position of the j^{th} particle. This interaction Hamiltonian is nothing else but the rotating-wave and long-wavelength approximation of the Hamiltonian $(-\sum_j P_j \cdot E)$.

In order to make the description of the material system complete, we must specify the initial state in which it is before being coupled to the field mode, that is, the equilibrium state which it reaches through interaction with the material reservoir alone. We shall assume that in this equilibrium state the various particles j are independent of each other, and the special level-pair of particle j that will interact with the field mode has a probability ρ_+ of being in the upper state and a probability ρ_- of being in the lower state. (These ρ are assumed to be independent of j .) We shall introduce

$$\beta_m = \rho_-/(\rho_+ - \rho_-); \langle p_j(0) \rangle = \rho_+ - \rho_- \quad (22)$$

so that

$$\rho_+ = \langle p_j(0) \rangle (1 + \beta_m); \rho_- = \langle p_j(0) \rangle \beta_m. \quad (23)$$

2.4 LOSS SYSTEM

The loss system will consist of an infinite number of harmonic oscillators, continuously, uniformly, and symmetrically distributed around ω_0 over an infinitely wide bandwidth, with a density σ_L per unit angular frequency (Fig. 2a). A particular harmonic

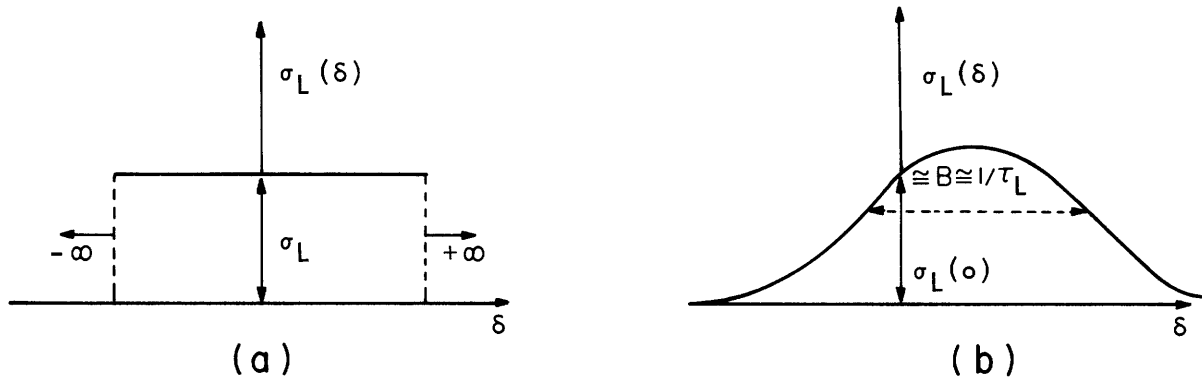


Fig. 2. Spectrum of loss system. (a) Uniform spectrum. (b) Nonuniform spectrum.

oscillator will be denoted by the index j , and its resonance angular frequency by ω_j . If we eliminate natural time dependence, it will be useful to introduce

$$\delta_j = \omega_j - \omega_0. \quad (24)$$

The assumption of a continuous spectrum of the loss system allows us to set

$$\sum_j \dots = \int \sigma_L d\delta \dots, \quad (25)$$

where the integral is over the whole spectrum. A particularly important summation is $\sum_j \exp(i\delta_j t)$. The assumption of a uniform and symmetrical spectrum over an infinitely wide bandwidth around ω_0 leads to

$$\sum_j e^{i\delta_j t} = 2\pi\sigma_L \delta(t), \quad (26)$$

where $\delta(t)$ is the Dirac delta function, with the property

$$\int_0^\epsilon \delta(t) dt = \int_{-\epsilon}^0 \delta(t) dt = \frac{1}{2}, \quad (27)$$

where ϵ is an infinitely small positive number.

If the operators b_j^+ and b_j are the creation and annihilation operators of the j^{th} harmonic oscillator, then the Hamiltonian of the loss system is given by

$$H_L = \sum_j \hbar\omega_j b_j^+ b_j, \quad (28)$$

with

$$[b_j, b_j^+] = \delta_{jj}, [b_j, b_j] = [b_j^+, b_j^+] = 0. \quad (29)$$

We shall assume that the coupling of the loss system to the field mode is described by the interaction Hamiltonian

$$H_{fL} = \sum_j \hbar\lambda_j (a^+ b_j + b_j^+ a), \quad (30)$$

where λ_j is a real number independent of j . Louisell has accepted this Hamiltonian to describe coupling of the field mode to spin waves inside the cavity.¹ In the thesis on which this report is based, we have shown that the coupling of the cavity mode to the outside space is also described by this interaction Hamiltonian; the operators b_j, b_j^+ are then the amplitudes of an artificially closed outside-space cavity. We shall assume that the loss system consists only of outside-space modes.

In order to make the description of the loss system complete, we must specify the initial state in which it is launched at $t = 0$. We shall suppose that this initial state is the thermal equilibrium state at temperature T_L . We introduce

$$\beta_L = [\exp(\hbar\omega_O/kT_L)-1]^{-1} = \langle b_j^\dagger(O)b_j(O) \rangle \quad (31)$$

$$\langle b_j(O)b_j^\dagger(O) \rangle = 1 + \beta_L. \quad (32)$$

For reasons of completeness, we mention the implications of a continuous, non-uniform and unsymmetrical spectrum with a wide but finite bandwidth, $B = 1/\tau_L$ (Fig. 2b). If $\sigma_L(\delta)$ is the density function, then

$$\sum_j e^{i\delta_j t} = \int \sigma_L(\delta) e^{i\delta t} d\delta = 2\pi u_L(t), \quad (33a)$$

where $u_L(t)$ is a function sharply peaked around $t = 0$, with

$$\begin{aligned} \int_0^{\tau_L} u_L(t) dt &= \frac{1}{2}\sigma_L(O) + \frac{i}{2\pi} \int \mathcal{P} \left[\frac{\sigma(\delta)}{\delta} \right] d\delta \\ \int_{-\tau_L}^0 u_L(t) dt &= \frac{1}{2}\sigma_L(O) - \frac{i}{2\pi} \int \mathcal{P} \left[\frac{\sigma(\delta)}{\delta} \right] d\delta \\ \int_{-\tau_L}^{\tau_L} u_L(t) dt &= \sigma_L(O). \end{aligned} \quad (33b)$$

These remarks will suffice to extend some of our derivations to the case in which the loss reservoir introduces also frequency shifts.

2.5 MATERIAL RESERVOIR

As we have said, the coupling of the material reservoir to the material system causes pumping, nonradiative decay and randomization of the material system. These effects will usually be described in a phenomenological way. For instance, in Section III the effect of the material reservoir is taken into account through a single approximation in the equations of motion. In Section V the effect of the reservoir is taken into account by assuming that the particle j interacts freely with the field mode for some time t_j , is then taken out of the system and replaced by a new particle whose initial state is specified. Only in Section VI shall we adopt, by way of an example, a specific model for the reservoir and its coupling to the material system. We shall assume that each level pair of each particle j is coupled to its own set of harmonic oscillators that are distributed around the resonance frequency of that level pair in the same way as that discussed in section 2.4. At launching time the harmonic oscillators are in an equilibrium state (which is not necessarily the thermal equilibrium state).

Because of various treatments of the material reservoir in this report, we shall not give more details now. In general, the Hamiltonian of the material reservoir and its

interaction Hamiltonian with the material system will be denoted by H_R and H_{mR} .

2.6 HAMILTONIAN OF THE INTERACTING SYSTEMS

We shall summarize the results for the four interacting systems (Fig. 3): the field mode (a, a^\dagger); the material system consisting of N many-level systems (particles) of which

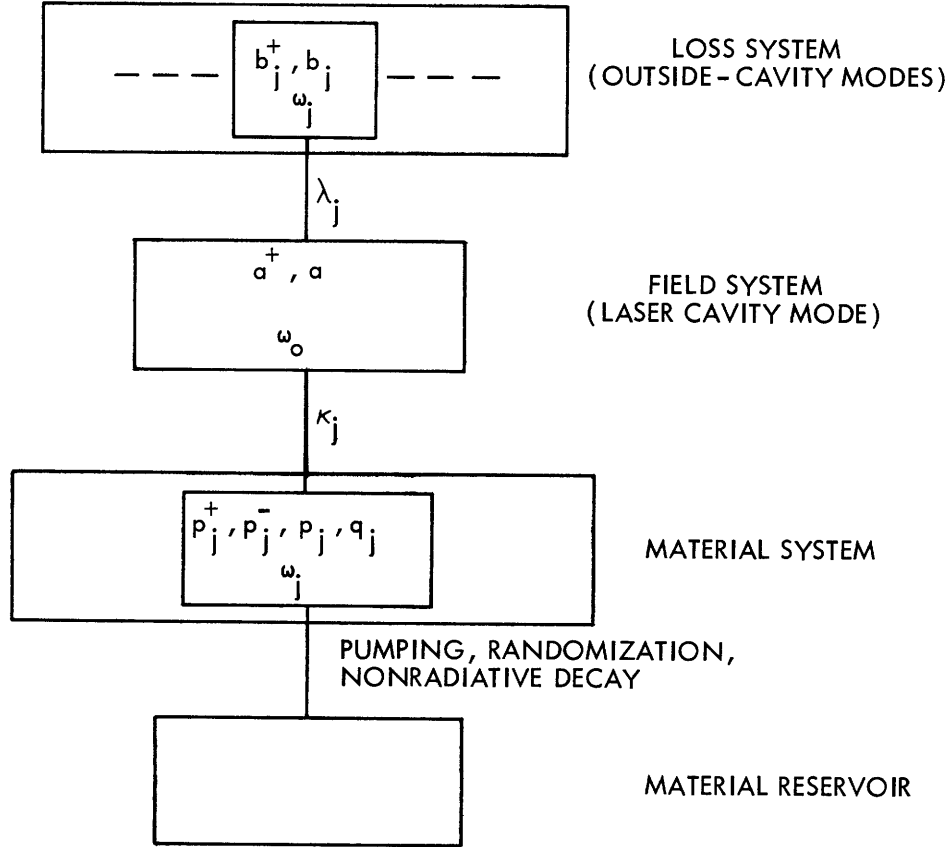


Fig. 3. Interacting systems.

two levels of each particle j (p_j^+, p_j^-, p_j, q_j) interact with the field mode; the set of harmonic oscillators (b_j, b_j^+) of the loss system (modes of the outside space); and material reservoir. The Hamiltonian of the total system is given by

$$\begin{aligned}
 H = & \hbar\omega_0 a^\dagger a + \sum_j (\hbar\Omega_i s_{ii})_j + \sum_j \hbar\omega_j b_j^\dagger b_j + H_R \\
 & + \sum_j i\hbar\kappa_j (a^\dagger p_j^- - p_j^+ a) + \sum_j \hbar\lambda_j (a^\dagger b_j + b_j^\dagger a) + H_{mR}.
 \end{aligned} \tag{34}$$

The properties of all these operators are described by Eq. 11, 13, 17, and 29. If we disregard for the moment the effect of the material reservoir, we obtain for the Heisenberg equations of motion

$$\frac{da(t)}{dt} = -i\omega_o a(t) + \sum_j \kappa_j p_j^-(t) - \sum_j i\lambda_j b_j(t); \quad \text{and Hermitian conjugate} \quad (35)$$

$$\frac{dp_j^-(t)}{dt} = -i\omega_j p_j^-(t) + p_j(t) a(t); \quad \text{and Hermitian conjugate} \quad (36)$$

$$\frac{dp_j(t)}{dt} = -2\kappa_j \left[p_j^+(t) a(t) + a^+(t) p_j^-(t) \right] \quad (37)$$

$$\frac{dq_j(t)}{dt} = 0 \quad (38)$$

$$\frac{db_j(t)}{dt} = -i\omega_j b_j(t) - i\lambda_j a(t); \quad \text{and Hermitian conjugate.} \quad (39)$$

These equations constitute the basis of subsequent sections.

III. OPERATOR NOISE SOURCES FOR THE INHOMOGENEOUSLY BROADENED LASER BELOW THRESHOLD

We have already explained why we expect that for some quantum problems a technique should exist that is analogous to the classical noise-source technique. We shall now actually prove the existence of such a technique for a simple laser model operating below threshold. We shall introduce the concept of operator noise source. We shall restrict ourselves to first-order and second-order moments; in Section IV we shall extend the operator noise-source technique to higher order moments.

3.1 LASER MODEL

We describe a laser model by making the following assumptions about the three essential systems: (i) the field oscillates in a single mode of the closed laser cavity with resonance angular frequency ω_0 ; (ii) the material system is inhomogeneously broadened with a frequency-independent line shape, that is, the two-level systems have resonance frequencies that are distributed symmetrically and uniformly around ω_0 over an infinitely large range; (iii) the loss system also has a frequency-independent line shape. The inversion of the material and the loss are such that the laser operates below threshold. We assume also that there are so many harmonic oscillators in the loss system and so many two-level systems in the material systems that summations over the index j (over the individual oscillators) can always be replaced by integrals over their resonance angular frequencies ω_j . Finally, we assume that the coupling constants κ_j and λ_j are independent of j , so that they can be denoted κ and λ . We shall mention the conditions under which a less idealized laser model will lead approximately to the same results as this idealized model.

This model is described by Eqs. 35-39. The laser mode has a resonance angular frequency ω_0 . The resonance angular frequencies ω_j cover an infinitely large range. We introduce

$$\delta_j = \omega_j - \omega_0 \quad (40)$$

which has the range $(-\infty, +\infty)$. We first eliminate the natural time dependence of the operators, that is, we make the substitutions

$$a(t) \rightarrow a(t) e^{-i\omega_0 t}; p_j^-(t) \rightarrow p_j^-(t) e^{-i\omega_j t}; b_j^-(t) \rightarrow b_j^-(t) e^{-i\omega_j t}; \quad \text{and Hermitian conjugate.} \quad (41)$$

Note that we retain the same notation for the new, slowly time-variant operators. The equations of motion for these new operators are

$$(d/dt)[a(t)] = \sum_j \kappa_j p_j^-(t) e^{-i\delta_j t} - \sum_j i\lambda_j b_j^-(t) e^{-i\delta_j t}; \quad \text{and Hermitian conjugate} \quad (42)$$

$$(d/dt)[p_j^-(t)] = \kappa_j p_j(t) a(t) e^{i\delta_j t}; \quad \text{and Hermitian conjugate} \quad (43)$$

$$(d/dt)[p_j(t)] = -2\kappa_j \left[p_j^+(t) a(t) e^{i\delta_j t} + a^+(t) p_j^-(t) e^{-i\delta_j t} \right] \quad (44)$$

$$(d/dt)[b_j^-(t)] = -i\lambda_j a(t) e^{i\delta_j t}; \quad \text{and Hermitian conjugate.} \quad (45)$$

The system is launched at $t = 0$. At that time each two-level system of the material system is assumed to be in an equilibrium state characterized by a probability ρ_+ of being in the upper state and a probability ρ_- of being in the lower state. We have

$$\begin{aligned} \langle p_j(0) \rangle &= \rho_+ - \rho_-; \quad \left\langle \left[p_j^-(0) \right]^r \right\rangle = \left\langle \left[p_j^+(0) \right]^r \right\rangle = 0, \quad (r=1, 2) \\ \langle p_j^+(0) p_j^-(0) \rangle &= \rho_+ = \langle p_j(0) \rangle (1 + \beta_m) \\ \langle p_j^-(0) p_j^+(0) \rangle &= \rho_- = \langle p_j(0) \rangle \beta_m \\ \beta_m &= [(\rho_+/\rho_-) - 1]^{-1}. \end{aligned} \quad (46)$$

At launching time $t = 0$, the loss system is assumed to be in thermal equilibrium at temperature T_L . We have

$$\begin{aligned} \left\langle \left[b_j^-(0) \right]^r \right\rangle &= \left\langle \left[b_j^+(0) \right]^r \right\rangle = 0, \quad (r=1, 2) \\ \left\langle b_j^+(0) b_j^-(0) \right\rangle &= [\exp(\hbar\omega_0/kT_L) - 1]^{-1} = \beta_L, \quad (\omega_j \approx \omega_0) \\ \left\langle b_j^-(0) b_j^+(0) \right\rangle &= 1 + \beta_L. \end{aligned} \quad (47)$$

Different two-level systems or harmonic oscillators are independent at $t = 0$.

It is generally agreed that for operation points "sufficiently" below threshold, one may replace the time-dependent operator $p_j(t)$ in Eq. 43 by the constant c-number $\langle p_j(0) \rangle$. We can then omit Eq. 44, while Eqs. 42, 43, and 45 become a set of linear operator equations. They have been solved by other authors.¹ It is not our task to justify this approximation. We shall adhere to it, but solve these linear operator equations by means of the noise-source formalism without making additional approximations. Our technique is thus as approximate as those previously given.¹

3.2 OPERATOR NOISE-SOURCE FORMULATION

We replace the differential equations 43 and 45 by the integral equations

$$\begin{aligned}
p_j^-(t) &= p_j^-(0) + \int_0^t \kappa_j \langle p_j(0) \rangle a(t') e^{i\delta_j t'} dt'; \quad \text{and Hermitian conjugate} \\
b_j^-(t) &= b_j^-(0) - \int_0^t i\lambda_j a(t') e^{i\delta_j t'} dt'; \quad \text{and Hermitian conjugate.}
\end{aligned} \tag{48a}$$

Equation 42 can thus be replaced by

$$\begin{aligned}
(d/dt)[a(t)] &= \sum_j \kappa_j p_j^-(0) e^{-i\delta_j t} - \sum_j i\lambda_j b_j^-(0) e^{-i\delta_j t} + \int_0^t \sum_j \kappa_j^2 \langle p_j(0) \rangle e^{-i\delta_j(t-t')} \\
&\quad \times a(t') dt' - \int_0^t \sum_j \lambda_j^2 e^{-i\delta_j(t-t')} a(t') dt'; \quad \text{and Hermitian conjugate.}
\end{aligned} \tag{48b}$$

We now use the assumptions that the summations over j can be replaced by integrals over δ . Thus

$$\sum_j \dots \int_{-\infty}^{+\infty} (\sigma_L \text{ or } \sigma_m) d\delta \dots, \tag{49}$$

and that the line shapes $\kappa^2 \langle p(0) \rangle \sigma_m$ and $\lambda^2 \sigma_L$ are independent of δ . The quantities σ_L and σ_m are the density of oscillators per unit angular frequency for the loss system and the material system. The last two terms on the right-hand side of Eq. 48b can then be written

$$\begin{aligned}
&\int_0^t a(t') dt' \int_{-\infty}^{+\infty} \kappa^2 \langle p(0) \rangle \sigma_m e^{-i\delta(t-t')} d\delta \\
& - \int_0^t a(t') dt' \int_{-\infty}^{+\infty} \lambda^2 \sigma_L e^{-i\delta(t-t')} d\delta.
\end{aligned} \tag{50}$$

Using the properties of the Dirac delta function $\delta(t)$ (do not confuse it with the angular frequency δ), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \kappa^2 \langle p(0) \rangle \sigma_m e^{-i\delta(t-t')} d\delta &= \kappa^2 \langle p(0) \rangle \sigma_m 2\pi\delta(t-t') = 2\gamma\delta(t-t') \\
\int_{-\infty}^{\infty} \lambda^2 \sigma_L e^{-i\delta(t-t')} d\delta &= \lambda^2 \sigma_L 2\pi\delta(t-t') = 2\mu\delta(t-t'),
\end{aligned} \tag{51}$$

where the Dirac delta function $\delta(t)$ must be interpreted as a completely symmetric function of t . That is,

$$\int_0^\epsilon \delta(t) dt = 1/2; \quad \int_{-\epsilon}^\epsilon \delta(t) dt = 1.$$

From Eqs. 48b, 50 and 51 we finally obtain

$$\begin{aligned} (d/dt)[a(t)] + (\mu - \gamma) a(t) &= x^-(t) \\ (d/dt)[a^+(t)] + (\mu - \gamma) a^+(t) &= x^+(t), \end{aligned} \quad (52)$$

where

$$\begin{aligned} x^-(t) &= \sum_j \kappa_j p_j^-(0) e^{-i\delta_j t} + \sum_j (-i) \lambda_j b_j^-(0) e^{-i\delta_j t} = x_m^-(t) + x_L^-(t) \\ x^+(t) &= \sum_j \kappa_j p_j^+(0) e^{i\delta_j t} + \sum_j i\lambda_j b_j^+(0) e^{i\delta_j t} = x_m^+(t) + x_L^+(t). \end{aligned} \quad (53)$$

The operator equations (52) have been obtained from the original equations (42), (43), and (47) by eliminating the unknown variables of the material and loss systems at t . The operators $x^\pm(t)$ are expressed in terms of material-system and loss-system variables that refer to states with known properties. This has the following consequences. First, the moments of $x^\pm(t)$ in principle can be calculated. Second, the first-order moments of $x^\pm(t)$ in particular are zero. If we thus should set the left-hand side of Eqs. 52 equal to zero, we would obtain the form of the equations for the first-order moments $\langle a(t) \rangle$ and $\langle a^+(t) \rangle$. Third, the second-order moments of $x^\pm(t)$ are not all zero. Therefore, the second-order moments of $a(t)$ and $a^+(t)$ will obey equations that are different from the equations of the first-order moments, because of the presence of $x^-(t)$ and $x^+(t)$ in Eqs. 52.

Equations 52 have all of these characteristics in common with the equations for classical systems driven by Langevin noise sources. We shall therefore call $x^-(t)$ and $x^+(t)$ "operator Langevin noise sources," or simply "operator noise sources."

3.3 SOLUTION FOR FIRST-ORDER AND SECOND-ORDER MOMENTS

We shall now solve Eqs. 52 for the first-order and second-order moments of the field variables $a(t)$ and $a^+(t)$, in the steady state of the laser oscillator below threshold ($\mu > \gamma$). We use a technique, developed in Appendix A, which is formally analogous to the techniques used to solve classical linear equations driven by noise sources with known moments. This technique leads to unique solutions if the system is stable and second-order moments stay finite for the time difference going to infinity. We first need the moments of the noise sources. From Eqs. 46, 47, and 53 we obtain

$$\begin{aligned}
\langle x_L^-(t) \rangle &= \langle x_m^-(t) \rangle = \langle [x_L^-(t)]^2 \rangle = \langle [x_m^-(t)]^2 \rangle = 0; \quad \text{and Hermitian conjugate} \\
\langle x_L^+(t+\tau) x_L^-(t) \rangle &= \sum_j \lambda_j^2 \langle b_j^+(0) b_j^-(0) \rangle e^{i\delta_j \tau} \\
\langle x_m^+(t+\tau) x_m^-(t) \rangle &= \sum_j \kappa_j^2 \langle p_j^+(0) p_j^-(0) \rangle e^{i\delta_j \tau}.
\end{aligned} \tag{54}$$

Again replacing \sum_j by $\int \sigma d\delta$ and using the assumption of frequency-independent line shape, we obtain

$$\langle x_L^+(t+\tau) x_L^-(t) \rangle = 2\mu\beta_L \delta(\tau). \tag{55a}$$

In the same way, we prove

$$\langle x_L^-(t+\tau) x_L^+(t) \rangle = 2\mu(1+\beta_L) \delta(\tau) \tag{55b}$$

$$\langle x_m^+(t+\tau) x_m^-(t) \rangle = 2\gamma(1+\beta_m) \delta(\tau) \tag{55c}$$

$$\langle x_m^-(t+\tau) x_m^+(t) \rangle = 2\gamma\beta_m \delta(\tau). \tag{55d}$$

Finally, the material-noise sources are uncorrelated with the loss-noise sources, and therefore

$$\begin{aligned}
\langle x^+(t+\tau) x^-(t) \rangle &= [2\gamma(1+\beta_m) + 2\mu\beta_L] \delta(\tau) \\
\langle x^-(t+\tau) x^+(t) \rangle &= [2\gamma\beta_m + 2\mu(1+\beta_L)] \delta(\tau) \\
\langle x^-(t+\tau) x^+(t) - x^+(t) x^-(t+\tau) \rangle &= 2(\mu - \gamma) \delta(\tau).
\end{aligned} \tag{56}$$

The noise sources thus have a "white" spectrum. This feature and the deterministic character of μ and γ are consequences of the frequency-independent line shapes of the material and loss systems. We use the following notation

$$\begin{aligned}
\phi_{uv}(\tau) &= \langle u(t+\tau) v(t) \rangle \\
\phi[u, v](\tau) &= \langle u(t+\tau) v(t) - v(t) u(t+\tau) \rangle,
\end{aligned} \tag{57}$$

where u and v are arbitrary operators in a steady-state ensemble. We find from Eqs. 52 for the first-order moments in the steady state

$$\langle a(t) \rangle = \langle a^+(t) \rangle = 0. \tag{58}$$

To find the second-order moment $\phi_{a^+a}(\tau)$ we cross-multiply Eqs 52:

$$\left\langle \frac{da^+(t+\tau)}{dt} \frac{da(t)}{dt} \right\rangle + (\mu-\gamma) \left\langle a^+(t+\tau) \frac{da(t)}{dt} \right\rangle + (\mu-\gamma) \left\langle \frac{da^+(t+\tau)}{dt} a(t) \right\rangle + (\mu-\gamma)^2 \left\langle a^+(t+\tau) a(t) \right\rangle = \left\langle x^+(t+\tau) x^-(t) \right\rangle$$

and use the technique of Appendix A to obtain

$$-\ddot{\phi}_{a^+a}(\tau) + (\mu-\gamma)^2 \phi_{a^+a}(\tau) = \phi_{x^+x^-}(\tau).$$

The unique solution of this equation, which stays finite for $|\tau| \rightarrow \infty$, is

$$G_1(\tau) = \phi_{a^+a}(\tau) = \left[\frac{\gamma}{\mu-\gamma} (1+\beta_m) + \frac{\mu}{\mu-\gamma} \beta_L \right] e^{-(\mu-\gamma)|\tau|}. \quad (59a)$$

In an analogous way, we find

$$\phi_{aa^+}(\tau) = \left[\frac{\gamma}{\mu-\gamma} \beta_m + \frac{\mu}{\mu-\gamma} (1+\beta_L) \right] e^{-(\mu-\gamma)|\tau|} \quad (59b)$$

$$\phi_{[a, a^+]}(\tau) = \exp[-(\mu-\gamma)|\tau|] \quad (59c)$$

$$\phi_{aa}(\tau) = \phi_{a^+a^+}(\tau) = 0. \quad (59d)$$

From these moments one can derive other useful moments. If we define the voltage as

$$V(t) = iC_v \left[a(t) e^{-i\omega_o t} - a^+(t) e^{i\omega_o t} \right], \quad (60)$$

where C_v is an appropriate normalization constant, then we find

$$\begin{aligned} (1/2) \langle V(t+\tau) V(t) + V(t) V(t+\tau) \rangle &= C_v^2 \frac{1}{2} \left[\phi_{a^+a}(\tau) + \phi_{aa^+}(\tau) \right] \times \cos \omega_o \tau \\ &= C_v^2 \left[\frac{\gamma}{\mu-\gamma} \left(\frac{1}{2} + \beta_m \right) + \frac{\mu}{\mu-\gamma} \left(\frac{1}{2} + \beta_L \right) \right] e^{-(\mu-\gamma)|\tau|} \cos \omega_o \tau. \end{aligned} \quad (61)$$

Results (59) and (61) can be rederived from an equivalent classical model described by

$$-\frac{da^+_{r.v.}}{dt} + (\mu-\gamma)a^+_{r.v.} = x^+_{r.v.}; \quad \text{and complex conjugate,} \quad (62)$$

where the variables are now classical random variables (denoted r. v.) instead of operators, that is, all commutators are zero. These equations can again be solved by the techniques of Appendix A. The results for the field moments depend, of course, on the assumptions that we make for the noise-source moments. If we want to rederive the results (59a) we must postulate

$$\left\langle x_{r.v.}^+(t+\tau) x_{r.v.}^-(t) \right\rangle = \left\langle x_{r.v.}^-(t) x_{r.v.}^+(t+\tau) \right\rangle = [2\gamma(1+\beta_m) + 2\mu\beta_L] \delta(\tau). \quad (63)$$

This is the so-called linear power noise source. If we want to rederive results (61) we must postulate

$$\left\langle x_{r.v.}^+(t+\tau) x_{r.v.}^-(t) \right\rangle = \left\langle x_{r.v.}^-(t) x_{r.v.}^+(t+\tau) \right\rangle = \left[2\gamma\left(\frac{1}{2} + \beta_m\right) + 2\mu\left(\frac{1}{2} + \beta_L\right) \right] \delta(\tau). \quad (64)$$

This is the so-called linear voltage noise source.

We have now shown that the concept of "operator noise source" can be defined and leads to quantum techniques that are in step-by-step analogy with classical Langevin techniques. We have derived first-order and second-order moments of the field variables. We have shown that equivalent classical models can be constructed which are solved by classical Langevin techniques. Because these moments and these equivalent models have been derived elsewhere, we did not consider it necessary to discuss these results and these equivalent models here.

In our idealized laser model, we could replace $\sum_j \dots$ by $\int \sigma d\delta$, and the material and loss systems have frequency-independent line shapes. For a more realistic laser model, we may replace \sum_j by $\int \sigma d\delta$ in the material system if the inverse of the spacing in frequency space between two neighboring oscillators is larger than the characteristic time with which the pump restores the inversion; in the loss system, if this inverse spacing is larger than the time since the system was launched. In the thesis on which this report is based we have shown that if the loss system is outside space, this inverse spacing is infinitely large. Furthermore, the line shapes of the material and loss systems must only be much broader than the "cold"-cavity bandwidth $2\mu(>2\gamma)$. If these line shapes are symmetric with respect to the cavity resonance frequency ω_0 , then γ and μ remain real; otherwise they become complex, with the real and imaginary parts related by the Kramers-Kronig relations. The moments of the noise sources depend only on the real parts.

IV. GAUSSIAN PROPERTIES OF MOMENTS OF OPERATORS

4.1 INTRODUCTION

We have restricted ourselves, in Section III, to first- and second-order moments of the noise source and field operators. The techniques were analogous to the techniques used to derive correlation functions of the output of classical linear systems driven by classical noise sources. For such systems it is possible to calculate higher order moments of the output from the second-order moments, if the input is Gaussian because the output is also Gaussian. One may expect that a quantum analogue to this classical theorem exists. The first difficulty is that there is no definition for the Gaussian character of moments of operators. We shall first try to develop such a definition.

The experimentally important quantities of both quantum theory and classical random theory are the moments of variables. In classical theory these moments can be calculated from the probability distribution function or from the characteristic function. The expressions of these functions for Gaussian random variables are well known. The corresponding quantities in quantum mechanics are the density matrix and the characteristic function. Moments of operators A_1, A_2, \dots are calculated from the density matrix ρ by means of $\langle A_1 A_2 \dots \rangle = \text{tr}(\rho A_1 A_2 \dots)$. Such a calculation is not necessarily formally analogous to the calculation of moments from a classical probability distribution. (For certain cases such a formal analogy can be established, for example, in the case that ρ, A_1, A_2, \dots are diagonal in the same representation or in the case that $A_1 A_2 \dots$ is a normally ordered product of boson creation and annihilation operators and ρ has a Glauber p-representation.⁵) We shall not try, however, to establish a general formal relationship of this kind. The relationship between characteristic functions $\chi(\xi_1, \xi_2, \dots) = \langle \exp(i\xi_1 A_1 + i\xi_2 A_2 + \dots) \rangle$ and moments is complicated in quantum mechanics through the commutation relations between A_1, A_2, \dots . Again, we shall not try to establish a general formal relationship with a classical characteristic function.

In classical random theory it is also possible to define Gaussian variables by the relationship between their higher order moments and their second- and first-order moments. It is this characteristic property of Gaussian variables which we shall try to generalize to the quantum case. We can always restrict ourselves to operators with zero expectation value (if $\langle A \rangle$ is different from zero, we shall consider $A - \langle A \rangle$). Because the order of factors in a moment is important in quantum mechanics, we expect difficulties in trying to formulate this generalization. We have found it convenient first to generalize the central limit theorem to quantum mechanics because its result will suggest the appropriate definition of the characteristic Gaussian property of moments of operators.

The proof that we give for the generalized central limit theorem is almost identical to a similar proof for the classical case. The essential features of this proof and its results, and the only aspect in which it differs from the classical proof and results will be summarized in section 4.1.

Consider a set of operators $\{X_i\}$, each consisting of a sum of N-operators:

$$X_i = \sum_{j=1}^N x_{ij}. \quad (65)$$

Suppose that

$$[x_{ij}, x_{i'j'}] = [x_{ij}, x_{i'j}] \delta_{jj'}, \quad (66)$$

and that there exists an ensemble in which each operator x_{ij} has an expectation value equal to zero, and in which these operators are independent of each other for different j. Therefore

$$\langle X_i \rangle = 0 \quad (67)$$

$$\langle X_{i_1} X_{i_2} \rangle = \left\langle \sum_{j_1} x_{i_1 j_1} \sum_{j_2} x_{i_2 j_2} \right\rangle = \sum_j \langle x_{i_1 j} x_{i_2 j} \rangle. \quad (68)$$

We consider the fourth-order moment

$$\langle X_{i_1} X_{i_2} X_{i_3} X_{i_4} \rangle = \left\langle \sum_{j_1} x_{i_1 j_1} \sum_{j_2} x_{i_2 j_2} \sum_{j_3} x_{i_3 j_3} \sum_{j_4} x_{i_4 j_4} \right\rangle \quad (69)$$

The expression between brackets on the right-hand side of Eq. (69) consists of a sum of products. We can subdivide the N^4 terms of this sum into 4-classes.

(i) Terms with four different j. Their expectation value is zero because of the independence and zero expectation value assumption for the x_{ij} . There are $N(N-1)(N-2)(N-3)$ such terms.

(ii) Terms with two and only two identical j. These can be subdivided in two sub-classes: (a) terms with three different j. Their expectation value is zero because of the independence and zero expectation value assumption. There are $6N(N-1)(N-2)$ such terms. (b) terms with two different j (terms in which the j-indices occur in two pairs). The two elements of one pair have the same j, the two pairs have different j. An element of one pair commutes with and is independent of each element of the other pair; the elements of the same pair do not necessarily commute. This part of the sum can be written

$$\begin{aligned} & \sum_j \langle x_{i_1 j} x_{i_2 j} \rangle \sum_{j' \neq j} \langle x_{i_3 j'} x_{i_4 j'} \rangle + \sum_j \langle x_{i_1 j} x_{i_3 j} \rangle \sum_{j' \neq j} \langle x_{i_2 j'} x_{i_4 j'} \rangle \\ & + \sum_j \langle x_{i_1 j} x_{i_4 j} \rangle \sum_{j' \neq j} \langle x_{i_2 j'} x_{i_3 j'} \rangle. \end{aligned} \quad (70)$$

There are $3N(N-1)$ terms in this part of the sum.

(iii) Terms with three and only three identical j . The fourth factor of these terms makes their expectation value equal to zero. There are $4N(N-1)$ such terms.

(iv) Terms with four identical j . Their contribution to Eq. 69 can be written

$$\sum_j \langle x_{i_1 j} x_{i_2 j} x_{i_3 j} x_{i_4 j} \rangle. \quad (71)$$

There are N terms of this type. The identity

$$N^4 = N(N-1)(N-2)(N-3) + 6N(N-1)(N-2) + 3N(N-1) + 4N(N-1) + N, \quad (72)$$

checks that we have enumerated all of the terms of Eq. 69. The only terms that are different from zero are those of classes 2b and 4. These contributions can be rewritten

$$\begin{aligned} \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle &= \sum_j \langle x_{i_1 j} x_{i_2 j} \rangle \sum_j \langle x_{i_3 j} x_{i_4 j} \rangle \\ &+ \sum_j \langle x_{i_1 j} x_{i_3 j} \rangle \sum_j \langle x_{i_2 j} x_{i_4 j} \rangle + \sum_j \langle x_{i_1 j} x_{i_4 j} \rangle \sum_j \langle x_{i_2 j} x_{i_3 j} \rangle \\ &+ \sum_j \left[\langle x_{i_1 j} x_{i_2 j} x_{i_3 j} x_{i_4 j} \rangle - \langle x_{i_1 j} x_{i_2 j} \rangle \langle x_{i_3 j} x_{i_4 j} \rangle \right. \\ &\left. - \langle x_{i_1 j} x_{i_3 j} \rangle \langle x_{i_2 j} x_{i_4 j} \rangle - \langle x_{i_1 j} x_{i_4 j} \rangle \langle x_{i_2 j} x_{i_3 j} \rangle \right]. \end{aligned} \quad (73)$$

Using Eq. 68, we find

$$\begin{aligned} \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle &= \langle x_{i_1} x_{i_2} \rangle \langle x_{i_3} x_{i_4} \rangle + \langle x_{i_1} x_{i_3} \rangle \langle x_{i_2} x_{i_4} \rangle \\ &+ \langle x_{i_1} x_{i_4} \rangle \langle x_{i_2} x_{i_3} \rangle + \sum_j \left[\langle x_{i_1 j} x_{i_2 j} x_{i_3 j} x_{i_4 j} \rangle - \langle x_{i_1 j} x_{i_2 j} \rangle \langle x_{i_3 j} x_{i_4 j} \rangle \right. \\ &\left. - \langle x_{i_1 j} x_{i_3 j} \rangle \langle x_{i_2 j} x_{i_4 j} \rangle - \langle x_{i_1 j} x_{i_4 j} \rangle \langle x_{i_2 j} x_{i_3 j} \rangle \right]. \end{aligned} \quad (74)$$

The result (74) is exact. We shall now investigate the consequences if N is very large. From (73), the first part of (74) consists of $3N^2$ terms, and the second of $4N$ terms. Thus far, we have made no other assumptions regarding the x_{ij} than the independence and zero expectation value assumption. We now make an additional order-of-magnitude assumption, that is, we assume that any typical second-order moment $\langle x_{i_1 j} x_{i_2 j} \rangle$ is of order Δx , and any typical n^{th} -order moment $\langle x_{i_1 j} x_{i_2 j} \dots x_{i_n j} \rangle$ is of

order $(\Delta x)^{n/2}$ or smaller. We have not assumed that all operators x_{ij} have identical moments for all possible j , but only that there is some simple order-of-magnitude relation between them. We then find that

$$\begin{aligned} \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = & \left[\langle x_{i_1} x_{i_2} \rangle \langle x_{i_3} x_{i_4} \rangle + \langle x_{i_1} x_{i_3} \rangle \langle x_{i_2} x_{i_4} \rangle \right. \\ & \left. + \langle x_{i_1} x_{i_4} \rangle \langle x_{i_2} x_{i_3} \rangle \right] [1+O(1/N)]. \end{aligned} \quad (75)$$

For a third-order moment, we obtain in a similar way

$$\begin{aligned} \langle x_{i_1} x_{i_2} x_{i_3} \rangle &= \sum_j \langle x_{i_1 j} x_{i_2 j} x_{i_3 j} \rangle = O[N(\Delta x)^{3/2}] \\ &= (\Delta x)^{3/2} O(1/\sqrt{N}), \end{aligned} \quad (76)$$

where Δx is the order of magnitude of a second-order moment of X_i . For an arbitrary even moment, we find

$$\langle x_{i_1} \dots x_{i_a} \dots x_{i_\beta} \dots x_{i_n} \rangle = \left[\sum \prod \langle x_{i_a} x_{i_\beta} \rangle \right] [1+O(1/N)], \quad (77)$$

where Π runs over a particular choice of pairs ($n/2$ factors) and Σ runs over all possible choices of pairs [$(n-1)(n-3)\dots 0 \cdot 1$ terms]. For an arbitrary odd moment we find

$$\langle x_{i_1} \dots x_{i_n} \rangle = (\Delta x)^{n/2} O(1/\sqrt{N}). \quad (78)$$

We summarize the essential features of the proof and the result for an n^{th} -order even moment. The only feature in which the result (77) differs from the analogous classical result is that the variables x_{i_a} and x_{i_β} when paired in the right-hand side of Eq. 77, appear in the same order as in the left-hand side of Eq. 77. The main argument of the proof is as follows. An n^{th} -order even moment consists of the sum of N^n terms. These terms are grouped in classes, with one and only one identical j , with two and only two identical j , with three and only three identical j , and so forth. In these classes we distinguish the subclasses that are zero because of the independence and zero expectation value assumptions regarding the x_{ij} . The most important remaining subclass is the one in which each j is paired to one other identical j . Because of the commutator assumption (4.2) we can rearrange the factors in each term so that pairs become neighboring elements, but one cannot change the order of the two elements of one pair; this is the only aspect in which our proof differs from a classical proof. All other steps are exactly the same as in the classical proof, in particular: (i) $\sum \Pi \langle x_{i_a} x_{i_\beta} \rangle$ contains all of the

$\left[(n-1)(n-3)\dots 1 \cdot N(N-1)\dots (N-\frac{n}{2}+1) \right]$ terms of this subclass and $\left\{ (n-1)(n-3)\dots \cdot 1 \left[N^{n/2} - N(N-1)\dots (N-\frac{n}{2}+1) \right] \right\}$ additional ones; the difference is clearly of order $(1/N)$ of $\sum \prod \langle X_{i_a} X_{i_\beta} \rangle$; (ii) all other subgroups have $(1/N)$ or $(1/N^2)$ terms less than $\sum \prod \langle X_{i_a} X_{i_\beta} \rangle$.

4.2 DEFINITIONS

We say that the moments of a set of operators $\{X_i\}$ with zero mean have the "Gaussian characteristic property" if for all even moments of order $n(n=2, 4, \dots)$

$$\langle X_{i_1} \dots X_{i_a} \dots X_{i_\beta} \dots X_{i_n} \rangle = \sum \prod \langle X_{i_a} X_{i_\beta} \rangle \quad (79)$$

and if for all odd moments of order $n(n=1, 3, \dots)$

$$\langle X_{i_1} \dots X_{i_n} \rangle = 0, \quad (80)$$

where Π runs over a particular choice of pairs ($n/2$ factors) and Σ runs over all possible choices of pairs $[(n-1)(n-3)\dots \cdot 1 \text{ terms}]$. Note that a particular pair in the right-hand side of (79) occurs in the same order as in the left-hand side of (79).

If the operators X_i depend on a parameter t , we call them operator processes $X_i(t)$. The moments of operator processes with zero mean have the Gaussian characteristic property if

$$\begin{aligned} \langle X_{i_1}(t_1) \dots X_{i_a}(t_a) \dots X_{i_\beta}(t_\beta) \dots X_{i_n}(t_n) \rangle &= \sum \prod \langle X_{i_a}(t_a) X_{i_\beta}(t_\beta) \rangle \\ \langle X_{i_1}(t_1) \dots X_{i_n}(t_n) \rangle &= 0 \end{aligned} \quad (81)$$

for n even and n odd, and for all possible choices of $i_1 \dots i_n$ and t_1, \dots, t_n .

These definitions reduce to the classical definition if the variables X_i or $X_i(t_i)$ commute for all i and t_i . Note that in the classical case the Gaussian moment expansion is "characteristic" for Gaussian variables with zero mean, that is, it can be used as a definition. In the classical case the order of factors in a moment has no importance.

We mention that the Gaussian moment expansion is a property of both the operators and the ensemble. We shall use as equivalent expressions: (i) the moments have the Gaussian property, and (ii) the operators are Gaussian in the ensemble.

4.3 THEOREM 1: HARMONIC OSCILLATOR IN THERMAL EQUILIBRIUM

The creation and annihilation operators of a harmonic oscillator are Gaussian in the thermal equilibrium ensemble.

Proof: The density matrix ρ and the Hamiltonian H are

$$\rho = \frac{\exp(-H/kT)}{\text{tr}[\exp(-H/kT)]}; \quad H = \hbar\omega a^\dagger a. \quad (82)$$

In the energy representation ρ has matrix elements

$$\langle n | \rho | n' \rangle = \frac{\exp(-\lambda n)}{\sum_{n=0}^{\infty} \exp(-\lambda n)} \delta_{nn'} = (1-e^{-\lambda}) e^{-\lambda n} \delta_{nn'}, \quad (83)$$

where

$$\lambda = (\hbar\omega/kT).$$

We consider operators consisting of a product of a certain number of factors a or a^\dagger . Such an operator with an unequal number of a 's and a^\dagger 's has only off-diagonal matrix elements and its expectation value is zero. The statement also implies that all odd moments are zero. Furthermore, all even moments with an unequal number of a 's and a^\dagger 's are zero, in particular, $\langle (a)^2 \rangle = \langle (a^\dagger)^2 \rangle = 0$. If we write out the expression (79) for these moments, we see that both sides of the equation are zero (each term of the right-hand side contains at least one factor $\langle (a)^2 \rangle$ or $\langle (a^\dagger)^2 \rangle$). These moments obey the Gaussian requirement (79). The only moments for which we still have to prove the Gaussian requirement are those with an equal number of a 's and a^\dagger 's. We thus restrict ourselves to such moments. The second-order moments are

$$\begin{aligned} \langle a^\dagger a \rangle &= (1-e^{-\lambda}) \sum_{n=0}^{\infty} n e^{-\lambda n} = (1-e^{-\lambda})(-d/d\lambda)(1-e^{-\lambda})^{-1} \\ &= (e^\lambda - 1)^{-1} = \beta \end{aligned} \quad (84)$$

$$\langle aa^\dagger \rangle = 1 + \beta. \quad (85)$$

Note that

$$(-d/d\lambda)\beta = \beta + \beta^2 \quad ; \quad e^{-\lambda} = \beta(1-e^{-\lambda}) \quad (86)$$

The Gaussian requirement (79) for the higher order moments will be proved in three steps.

Step 1: All normally ordered products obey (79). The proof uses the method of complete induction. First, we have

$$\begin{aligned}
\langle (a^+)^{r+1} (a)^{r+1} \rangle &= (1-e^{-\lambda}) \sum_{n=0}^{\infty} n(n-1)(n-2)\dots(n-r) e^{-\lambda n} \\
&= (1-e^{-\lambda}) (-d/d\lambda) (-1-d/d\lambda) (-2-d/d\lambda) \dots (-r-d/d\lambda) (1-e^{-\lambda})^{-1} \\
&= (1-e^{-\lambda}) (-r-d/d\lambda) (-d/d\lambda) (-1-d/d\lambda) \dots (-r+1-d/d\lambda) (1-e^{-\lambda})^{-1} \\
&= (-r-d/d\lambda) [(1-e^{-\lambda}) (-d/d\lambda) (-1-d/d\lambda) \dots (-r+1-d/d\lambda) (1-e^{-\lambda})^{-1}] \\
&\quad + e^{-\lambda} (-d/d\lambda) (-1-d/d\lambda) \dots (-r+1-d/d\lambda) (1-e^{-\lambda})^{-1}
\end{aligned}$$

and thus

$$\langle (a^+)^{r+1} (a)^{r+1} \rangle = (-r-d/d\lambda) \langle (a^+)^r (a)^r \rangle + \beta \langle (a^+)^r (a)^r \rangle.$$

Second, if

$$\langle (a^+)^r (a)^r \rangle = r! \beta^r,$$

then

$$\begin{aligned}
\langle (a^+)^{r+1} (a)^{r+1} \rangle &= r(r!) \beta^{r-1} (\beta + \beta^2) - r(r!) \beta^r + \beta(r!) \beta^r \\
&= (r+1)! \beta^{r+1}.
\end{aligned}$$

Third, since

$$\langle (a^+)^1 (a)^1 \rangle = 1! \beta^1$$

we have, by complete induction,

$$\langle (a^+)^r (a)^r \rangle = r! \beta^r = r! (\langle a^+ a \rangle)^r. \quad (87)$$

If we now write the right-hand side of (79) for the normally ordered product $(a^+)^r (a)^r$, then we find that all terms of the Gaussian moment expansion are equal for this case. Each term consists of a product of r factors $\langle a^+ a \rangle$ and is thus equal to $\langle a^+ a \rangle^r$; there are $r!$ such terms, that is, the number of pair choices $a^+ a$. The Gaussian moment expansion (79) leads also to (87).

Step 2: If the expectation value of one particularly ordered product A obeys the Gaussian moment expansion, and if the expectation value of the product B formed from A by dropping two neighboring creation and annihilation operators obeys also the Gaussian moment expansion, then the expectation value of the product C formed from A by commuting these two neighboring creation and annihilation operators also obeys the Gaussian moment expansion.

Indeed, from the Gaussian moment expansion of $\langle A \rangle$ and $\langle B \rangle$ follows

$$\langle A \rangle = \langle \dots a^+ a \dots \rangle = \langle a^+ a \rangle \langle B \rangle + \sum' \Pi' \langle a^+ a \text{ or } aa^+ \rangle,$$

where $\sum' \Pi'$ runs over all choices of pairs in which the particular a and a^+ under consideration are not paired; this $\sum' \Pi'$ expression is not affected by the order of these a and a^+ . For the operator C we have

$$\begin{aligned} \langle C \rangle &= \langle \dots aa^+ \dots \rangle = \langle A \rangle + \langle B \rangle \\ &= \langle a^+ a \rangle \langle B \rangle + \langle B \rangle + \sum' \Pi' \langle a^+ a \text{ or } aa^+ \rangle \\ &= \langle aa^+ \rangle \langle B \rangle + \sum' \Pi' \langle a^+ a \text{ or } aa^+ \rangle. \end{aligned}$$

The last equality expresses the Gaussian moment expansion of $\langle C \rangle$.

Step 3: From Steps 1 and 2 follows the theorem by complete induction. First, Step 1 proves the Gaussian moment expansion of all normally ordered products. Second, repeated application of Step 2 allows us first to prove the Gaussian moment expansion of all fourth-order moments, then of all sixth-order moments, and so forth.

We illustrate this theorem by a few examples:

$$\langle (a^+)^r (a)^r \rangle = r! \beta^r \quad (88)$$

$$\langle (a)^r (a^+)^r \rangle = r! (1+\beta)^r \quad (89)$$

$$\langle a^+ a a a^+ \rangle = \langle a^+ a \rangle \langle a a^+ \rangle + \langle a^+ a \rangle \langle a a^+ \rangle = 2\beta(1+\beta)$$

4.4 THEOREM 2: CENTRAL LIMIT THEOREM

We consider a set of operators $X_i(N, \lambda)$ that are the sum of N operators λx_{ij}

$$X_i(N, \lambda) = \sum_{j=1}^N \lambda x_{ij}. \quad (90)$$

We suppose that in an appropriate ensemble the operators x_{ij} have zero mean, are independent of each other for different j 's, and have n^{th} -order moments $\langle x_{i_a j} \dots x_{i_\beta j} \rangle$ all of the same order of magnitude $(\Delta x)^{n/2}$ or smaller. If we let N tend to infinity and λ to zero in such a way that

$$\langle X_{i_a} X_{i_\beta} \rangle = \sum_{j=1}^N \lambda^2 \langle x_{i_a j} x_{i_\beta j} \rangle$$

stays finite, then the higher order moments of X_i obey in the limit the Gaussian moment expansion (79) and (80).

This theorem follows immediately from Eqs. 77 and 78. It is a limit theorem. In any real case N is finite so that the X_i are only "approximately" Gaussian in the above-mentioned ensemble. We shall call these X_i Gaussian in that ensemble if they obey also Theorems 3 and 4.

4.5 THEOREM 3: LINEAR SUPERPOSITION

"A set of linear combinations of operators that are Gaussian in an appropriate ensemble, are Gaussian in that ensemble." This theorem is an immediate consequence of the fact that the expectation value of a sum of operators is the sum of the expectation values of the operators.

Proof: Consider the operators X_i that are Gaussian in an appropriate ensemble, and consider the linear combinations u_i :

$$u_i = \sum_j \lambda_{ij} X_j = \lambda_{ij} X_j.$$

We have

$$\begin{aligned} & \langle u_{i_1} \dots u_{i_a} \dots u_{i_\beta} \dots u_{i_n} \rangle \\ &= \langle \lambda_{i_1 j_1} X_{j_1} \dots \lambda_{i_a j_a} X_{j_a} \dots \lambda_{i_\beta j_\beta} X_{j_\beta} \dots \lambda_{i_n j_n} X_{j_n} \rangle \\ &= \lambda_{i_1 j_1} \dots \lambda_{i_n j_n} \langle X_{j_1} \dots X_{j_n} \rangle \\ &= \lambda_{i_1 j_1} \dots \lambda_{i_n j_n} \sum \prod \langle X_{j_a} X_{j_\beta} \rangle \\ &= \sum \prod \langle \lambda_{i_a j_a} X_{j_a} \lambda_{i_\beta j_\beta} X_{j_\beta} \rangle \\ &= \sum \prod \langle u_{i_a} u_{i_\beta} \rangle. \end{aligned}$$

Corollary. Integrals of operator processes that are Gaussian in an appropriate ensemble, are Gaussian in that ensemble.

If

$$u(t) = \int_{t_0}^t f(t') x(t') dt',$$

then $u(t)$ is in fact a linear combination of operators that are Gaussian in that ensemble.

4.6 THEOREM 4: GAUSSIAN OPERATORS DRIVING LINEAR SYSTEMS

The outputs of linear operator systems are operator processes whose moments have the characteristic Gaussian property, if the moments of the initial output operators and of the input operator processes have the characteristic Gaussian property. If the system is stable and if we disregard transient solutions, then only the moments of the input operator processes must have the Gaussian property.

Proof: We consider first a linear system of the type

$$(d/dt)[u(t)] + su(t) = n_u(t), \quad (91)$$

where s is a c-number, $u(t)$ is the output operator process, and $n_u(t)$ is the input operator process. If $u(0)$ is the initial output operator, then the differential equation (91) can be replaced by the integral equation

$$u(t) = u(0) e^{-st} + \int_0^t e^{-s(t-\tau)} n_u(\tau) d\tau. \quad (92)$$

The corollary of Theorem 3 proves the Gaussian character of the output process $u(t)$. If $\text{Re}(s) > 0$ (stable system) and if we restrict ourselves to times t where $\text{Re}(st) \gg 1$, we obtain

$$u(t) = \int_0^t e^{-s(t-\tau)} n_u(\tau) d\tau \quad (93)$$

which proves the second part of Theorem 4.

Second, we consider a linear system of the type

$$(d/dt)[x_i(t)] + a_{ij}x_j(t) = n_{x_i}(t) \quad (i, j = 1, 2, \dots, N), \quad (94)$$

where a_{ij} are c-numbers, and we have used the Einstein summation notation. By diagonalizing Eqs. 94, we can reduce them to N equations of type (91). Multiply each of the equations (94) with the numbers λ_{ki} and add. We obtain

$$(d/dt)(\lambda_{ki}x_i) + \lambda_{ki}a_{ij}x_j = \lambda_{ki}n_{x_i}. \quad (95)$$

The set of N homogeneous equations

$$\lambda_{ki}a_{ij} = s_k \lambda_{kj} \quad (j = 1, 2, \dots, N) \quad (96)$$

has a solution different from zero for each of the N roots $s_1, \dots, s_k, \dots, s_N$ of

$$\det[a_{ij} - s\delta_{ij}] = 0, \quad (97)$$

where the root s_k corresponds to the solution λ_{ki} ($i, k = 1, \dots, N$).

If we put

$$u_k = \lambda_{ki} x_i \quad ; \quad n_{u_k} = \lambda_{ki} n_{x_i}, \quad (98)$$

we obtain

$$(d/dt)u_k + s_k u_k = n_{u_k}. \quad (99)$$

From Eqs. 98 it follows that if $x_i(0)$ and $n_{x_i}(t)$ are joint Gaussian, then $u_k(0)$ and $n_{u_k}(t)$ are joint Gaussian. From Eqs. 99 it follows that $u_k(t)$ are joint Gaussian. Because $x_i = (\lambda^{-1})_{ik} u_k$, it follows that the $x_i(t)$ are joint Gaussian.

Third, we consider linear systems that involve higher order derivatives of the output processes. These can be easily reduced to systems of type (94) by appropriate substitutions. For instance, $\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = n$ is equivalent to $\frac{dx}{dt} - u = 0$, $\frac{du}{dt} + au + bx = n$.

4.7 APPLICATION: HIGHER ORDER MOMENTS OF THE FIELDS OF THE LASER OSCILLATOR BELOW THRESHOLD

Equations (52) show that the laser oscillator is a stable linear system driven by operator noise sources. These noise sources are Gaussian processes in the ensemble to which they refer. The noise sources $x_L^+(t)$ and $x_L^-(t)$ are Gaussian processes because they are linear combinations of the set of creation and annihilation operators $b_j^+(0)$ and $b_j^-(0)$; these refer to a thermal equilibrium ensemble and are independent of each other for different j . The noise sources $x_m^+(t)$ and $x_m^-(t)$ are Gaussian processes (and obey Theorem 4) if there is a sufficiently large number of two-level systems in the minimum bandwidth $d\delta$ of the measuring apparatus (note that the laser device acts as a linear filter of width $(\gamma - \mu)$ for these noise sources and that the laser field is presumably measured by means of devices that contain linear filters of even much narrower width). For the laser one may assume that this condition is fulfilled. Finally, the noise sources $x_L^\pm(t)$ and $x_m^\pm(t)$ are independent so that $x^+(t)$ and $x^-(t)$ are Gaussian processes.

According to Theorem 4, these Gaussian processes will drive the stable linear laser to a Gaussian steady state. All higher order moments of $a(t)$ and $a^+(t)$ can thus be derived from the second-order moments given in Eqs. 59. In particular,

$$\begin{aligned} G_2(a, \tau) &= \langle a^+(t) a^+(t+\tau) a(t+\tau) a(t) \rangle \\ &= \phi_{a^+ a}(-\tau) \phi_{a^+ a}(\tau) + \phi_{a^+ a}(0) \phi_{a^+ a}(0), \end{aligned} \quad (100)$$

where

$$G_1(a, 0) = \phi_{a^+ a}(0) = [\gamma(1 + \beta_m) + \mu\beta_L] / (\mu - \gamma). \quad (101)$$

For the relative photon number fluctuations we obtain

$$\frac{G_2(\tau) - [G_1(0)]^2}{G_1(0)} = \frac{1}{\mu - \gamma} [\gamma(1 + \beta_m) + \mu\beta_L] e^{-2(\mu - \gamma)|\tau|}. \quad (102)$$

This expression is identical with results derived by Haus² by means of equivalent classical circuits for the laser below threshold, or by others by means of quantum formalisms other than the operator noise-source formalism. In order to demonstrate the exact correspondence, we introduce the Freed and Haus notation. We set

$$2(\mu - \gamma) = \omega_o/Q' \quad ; \quad 2\mu = \omega_o/Q_o \quad ; \quad |2\gamma| = \omega_o/|Q_m^o|, \quad (103)$$

where Q' is the "hot" cavity Q , Q_o is the "cold" cavity Q , and $|Q_m^o|$ is the quality factor of the laser material when there is no saturation effect. We note that the average power transmitted in the laser beam is

$$\bar{p} = 2\mu\hbar\omega_o G_1(0) \quad (104)$$

and that the Fourier spectrum of $(1/a) \exp(-a|\tau|)$ is given by

$$(1/2\pi) \int_{-\infty}^{\infty} (1/a) e^{-a|\tau|} e^{-i\omega\tau} d\tau = [\pi(\omega^2 + a^2)]^{-1}. \quad (105)$$

Note, furthermore, that from Eq. 46 it follows that

$$1 + \beta_m = \rho_+ / (\rho_+ - \rho_-). \quad (106)$$

The quantities γ and β_m have always the same sign because γ is proportional to $\rho_+ - \rho_-$. We restrict ourselves to the case $\beta_L = 0$. If we call $\Phi_p(\omega)$ the Fourier spectrum of $(2\mu\hbar\omega_o)^2 \{G_2(\tau) - [G_1(0)]^2\}$, then we obtain from (102)

$$\frac{\Phi_p(\omega)}{\bar{p}} = (1/\pi) \left| \frac{\rho_+}{\rho_+ - \rho_-} \right| \frac{\hbar\omega_o}{Q_o |Q_m^o|} \frac{\omega^2}{\omega^2 + (\omega_o/Q')^2} \quad (107)$$

This expression should be compared with expression (20) of Freed and Haus.⁷

Another interesting moment is $\langle n(t+\tau)n(t) \rangle$ where n is the photon number operator $a^\dagger a$. We obtain

$$\langle a^\dagger(t+\tau)a(t+\tau)a^\dagger(t)a(t) \rangle = \Phi_{a^\dagger a}(\tau) \phi_{aa^\dagger}(\tau) + [G_1(0)]^2 \quad (108)$$

or by using the results (59a and b)

$$\frac{\langle n(t+\tau)n(t) \rangle - [G_1(0)]^2}{G_1(0)} = \frac{1}{\mu - \gamma} [\gamma\beta_m + \mu(1 + \beta_L)] e^{-2\mu - \gamma|\tau|}. \quad (109)$$

The difference between the right-hand side of Eqs. 109 and 102 is small (1 photon for

$\tau = 0$) but it nevertheless exists. Glauber⁵ and Kelley and Kleiner⁶ have shown that a photon-electron device actually measures the left-hand side of (102) and not the left-hand side of (109). In the equivalent circuit calculations of Haus^{2,7} it is not obvious whether one calculates the left-hand side of (102) or of (109). One obtains the result of the right-hand side of (102). The reason for this fortunate "coincidence" is the following. In the equivalent-circuit formalism one makes the statement that the classical noise source used to calculate G_1 is Gaussian. This immediately implies that all of the higher order moments that one calculates with the equivalent circuit will be the expectation values of the normally ordered product (the G-functions). Indeed, for a Gaussian field there is a $p(\alpha)$ representation⁵ that is formally identical with a classical Gaussian distribution, and with which one calculates expectation values of normally ordered products in exactly the same way as in the classical case. It is thus only for the normally ordered products (of higher order) that one obtains the right results in the equivalent classical circuit with a Gaussian classical noise source. We have shown that with the operator noise-source formalism one can easily obtain results for any product, normally ordered or not.

Other quantities of interest are the voltage $V(t)$ and the current $I(t)$:

$$\begin{aligned} V(t) &= C_V i \left[a(t) e^{-i\omega_0 t} - a^\dagger(t) e^{+i\omega_0 t} \right] \\ I(t) &= C_I \left[a(t) e^{-i\omega_0 t} + a^\dagger(t) e^{+i\omega_0 t} \right], \end{aligned} \quad (110)$$

where C_V and C_I are appropriate normalization constants. Theorem 3 shows that $V(t)$ and $I(t)$ are joint Gaussian processes in the steady state because they are linear combinations of the Gaussian processes $a(t)$ and $a^\dagger(t)$. We can again calculate all higher order moments from the second-order moments:

$$\begin{aligned} \langle V(t_1)V(t_2)V(t_3)V(t_4) \rangle &= \langle V(t_1)V(t_2) \rangle \langle V(t_3)V(t_4) \rangle \\ &\quad + \langle V(t_1)V(t_3) \rangle \langle V(t_2)V(t_4) \rangle + \langle V(t_1)V(t_4) \rangle \langle V(t_2)V(t_3) \rangle \\ \langle V(t_1)V(t_2)I(t_3)I(t_4) \rangle &= \langle V(t_1)V(t_2) \rangle \langle I(t_3)I(t_4) \rangle \\ &\quad + \langle V(t_1)I(t_3) \rangle \langle V(t_2)I(t_4) \rangle + \langle V(t_1)I(t_4) \rangle \langle V(t_2)I(t_3) \rangle. \end{aligned}$$

We shall not evaluate the actual values of these moments because their experimental significance has not yet been established.

4.8 PHYSICAL INTERPRETATION

We have calculated, by means of the operator noise-source formalism, various moments of the field variables in the steady state of the laser oscillator below threshold. We now want to discuss equivalent classical models. By "classical" we mean that both the system variables and the noise sources are classical random processes (all

commutators are zero); it does not mean that we have to explain all properties of the model itself (that is, of the equations of motion and of the moments of the noise sources) by classical arguments. By "equivalent" we mean that the moments of those system variables must be exactly equal to certain moments of the operator field variables calculated by direct quantum methods. These equivalent models do not constitute a "semi-classical" theory of the laser, if one means a theory of the laser in which the electromagnetic field is considered to be a classical field.

We shall consider two models. In the first model the system variables will describe a classical field obeying classical statistics; in the second, the system variable will be a number of classical particles that obey classical statistics. Since quantum mechanics is essentially a synthesis of the wave (or field) aspects and the quantum (or particle) aspects of nature, one cannot expect that a classical field theory alone, or a classical particle theory alone can explain all quantum-mechanical results. We shall indeed show that with the classical field model one can derive the expectation value of all normally-ordered products of the a, a^+ operators, that is, all of the G-functions. With the second model we shall be able to derive all the moments of the photon number operator, that is, all of the G' -functions. In both cases we shall prove these statements for first-order and second-order G-functions or photon number moments, and only mention the results for higher order moments.

4.8.1 Alternative Approach for Moments of the Photon Number Operator

We have derived second-order moments of the a, a^+ operators. In Section 4.7 we obtained fourth-order moments of these operators by exploiting the theorems about Gaussian operator processes. We could have calculated any higher order moment, in particular, all G-functions, and all moments of the photon number operator. We now want to give an alternative approach that allows us to obtain the first-order and second-order moments of the photon number operator. It seems a somewhat more complicated approach, but it will help us to construct and interpret the equivalent models.

From Eqs. 52 it follows

$$\frac{dn(t)}{dt} + 2gn(t) = a^+(t) x^-(t) + x^+(t) a(t), \quad (111)$$

where

$$g = \mu - \gamma \quad ; \quad n(t) = a^+(t) a(t). \quad (112)$$

Equation 111 is not of the Langevin type. It is indeed not split into a first term that contains only system variables (averaged equation) and a second term that is a noise source with zero average value. It is however (and so are also Eqs. 52) of the type discussed in Appendix C. There it is shown that such equations can be transformed to the Langevin form, the averaged equation can be found from the drift terms in the first-order term of a perturbation expansion, and the "xx" terms of the second-order term of the

perturbation expansion, and finally that the second-order moment of the Langevin noise source can be found from cross-multiplying the "x" terms of the first-order term of the perturbation expansion. If we apply the formulas (C.7), (C.9) and (C.13), we find from Eqs. 111 and 52 that Eq. 111 can be transformed to

$$\frac{dn(t)}{dt} + 2gn(t) = A + x(t), \quad (113)$$

with

$$A = \frac{1}{Dt} \int_t^{t+Dt} dt_1 \int_t^{t_1} dt_2 \left\langle x^+(t_2)x^-(t_1) + x^+(t_1)x^-(t_2) \right\rangle_R \quad (114a)$$

$$\langle x(t) \rangle = 0 \quad (114b)$$

$$\langle x(t+\tau)x(t) \rangle = \delta(\tau) \frac{1}{Dt} \int_t^{t+Dt} dt_1 \int_t^{t_1} dt_2 \langle \dots \rangle_R \quad (114c)$$

$$\langle \dots \rangle_R = \left\langle [a^+(t)x^-(t_1) + x^+(t_1)a(t)][a^+(t)x^-(t_2) + x^-(t_2)a(t)] \right\rangle_R.$$

Here, we must treat $a^\pm(t)$ in Eq. 114c as being independent of $x^\pm(t_1$ or $t_2)$, and $\langle \dots \rangle_R$ means an average over the initial state of the loss and material systems. If we use the results obtained in Eqs. 56 for the second-order moments of $x^\pm(t)$, we find immediately

$$\frac{dn(t)}{dt} + 2gn(t) = A + x(t) \quad (113)$$

with

$$g = \mu - \gamma \quad (115a)$$

$$A = 2\gamma(1+\beta_m) + 2\mu\beta_L \quad (115b)$$

$$\langle x(t) \rangle = 0 \quad (115c)$$

$$\langle x(t+\tau)x(t) \rangle_R = B(t)\delta(t) \quad (115d)$$

$$B(t) = 2\gamma(1+\beta_m)[\langle n(t) \rangle_R + 1] + 2\gamma\beta_m \langle n(t) \rangle_R \\ + 2\mu\beta_L[\langle n(t) \rangle_R + 1] + 2\mu(1+\beta_L) \langle n(t) \rangle_R \quad (115e)$$

$$\langle B(t) \rangle_F = B = 2\gamma(1+\beta_m)[\langle n(t) \rangle + 1] + 2\gamma\beta_m \langle n(t) \rangle \\ + 2\mu\beta_L[\langle n(t) \rangle + 1] + 2\mu(1+\beta_L) \langle n(t) \rangle. \quad (115f)$$

Note that $B(t)$ is an operator of the field system. Only if we take an additional average over the field state, we obtain the c-number B . In the steady state B is a constant.

From Eq. 113 it follows

$$\frac{d \langle n(t) \rangle}{dt} + 2g \langle n(t) \rangle = A, \quad (116)$$

so that the steady-state solution is

$$\langle n(t) \rangle = \frac{A}{2g}. \quad (117)$$

Using the techniques of Appendix A, we obtain from Eq. 113, in the steady state

$$\langle n(t+\tau)n(t) \rangle = \left(\frac{A}{2g} \right)^2 + \frac{B}{4g} e^{-2g|\tau|}. \quad (118)$$

It is easy to check that the results (117) and (118) agree with the results for $G_1(0)$ given in Eq. 59a and $G_2'(\tau)$ given in Eq. 100 or 102.

We want to interpret the constants A and B. If one accepts the statement that the spontaneous emission into a harmonic oscillator is in fact emission stimulated by the zero-point oscillations of this harmonic oscillator and is equal to the emission stimulated by one photon of this harmonic oscillator, then it is easy to understand the following statements. Since

$$2\gamma \langle n(t) \rangle = c_{(\rho_+ - \rho_-)}^{\text{st}} \langle n(t) \rangle = \text{net stimulated (photon) emission rate from the material system into the field mode,}$$

we have

$$2\gamma(1+\beta_m) = c_{\rho_+}^{\text{st}} = \text{spontaneous emission rate from material system into field mode.}$$

$$2\gamma(1+\beta_m)[1 + \langle n(t) \rangle] = \text{total emission rate from material system into field mode.}$$

$$2\gamma\beta_m \langle n(t) \rangle = c_{\rho_-}^{\text{st}} \langle n(t) \rangle = \text{stimulated (=total) absorption rate by material system out of field mode.}$$

Furthermore, since β_L is the average photon number per mode of the loss system, we have

$$2\mu\beta_L = \text{spontaneous emission rate from loss system into field mode.}$$

$$2\mu\beta_L \langle n(t) \rangle = \text{stimulated emission rate from loss system into field mode.}$$

$$2\mu\beta_L [1 + \langle n(t) \rangle] = \text{total emission rate from loss system into field mode.}$$

$$2\mu \langle n(t) \rangle = \text{spontaneous emission rate from field mode into loss system.}$$

$$2\mu \langle n(t) \rangle \beta_L = \text{stimulated emission rate from field mode into loss system.}$$

$$2\mu \langle n(t) \rangle (1+\beta_1) = \text{total emission rate from field mode into loss system.}$$

We have illustrated the effects of the loss system on the field system in Fig. 4. We

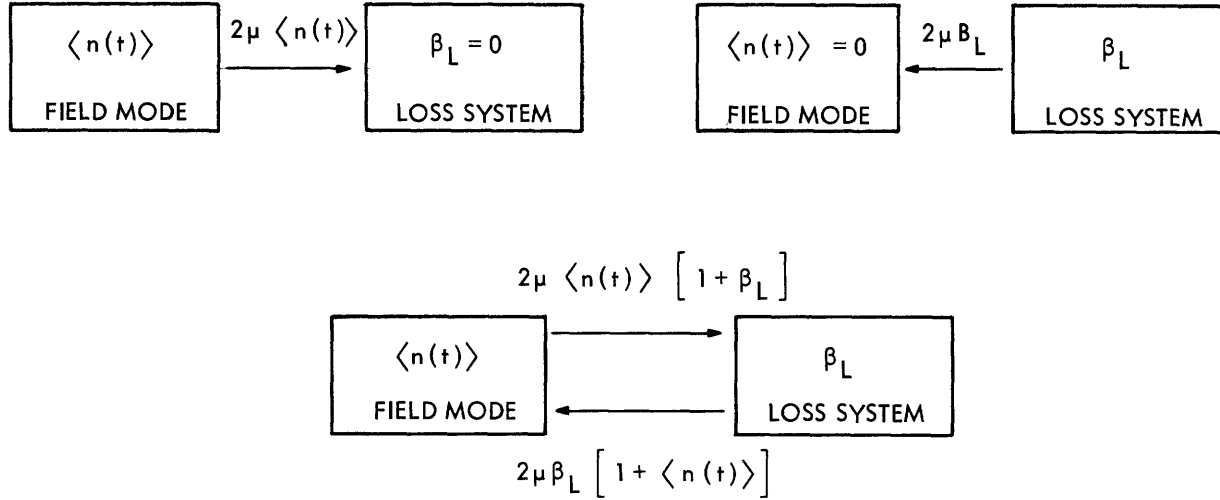


Fig. 4. Rates of photon number transfer from field mode to loss system.

conclude that

- A = total spontaneous emission rate into the field mode from both the material system and the loss system,
- = total rate or net rate if the field mode is not excited,
- B = total rate (rate in + rate out) into and out of the field mode from both the material system and the loss system.

We have illustrated these statements in Figs. 5 and 6.

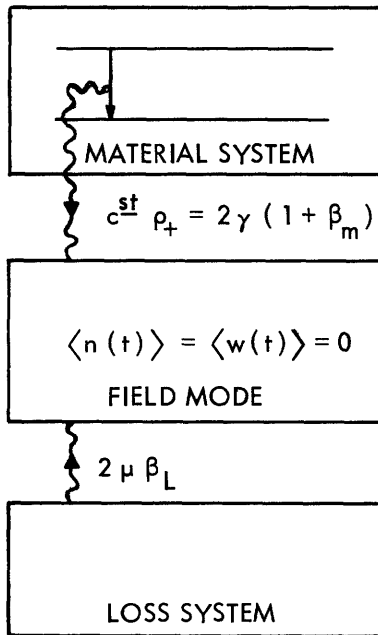


Fig. 5. Rates of energy transfer (in units $\hbar\omega_0$) into field mode from material system and loss system if field mode is not excited.

4.8.2 Model 1: Classical Field Model

We shall show that the results for $G_1(\tau)$ and $G_2(\tau)$ can be derived from an equivalent classical model in which the system variables describe a classical field driven by

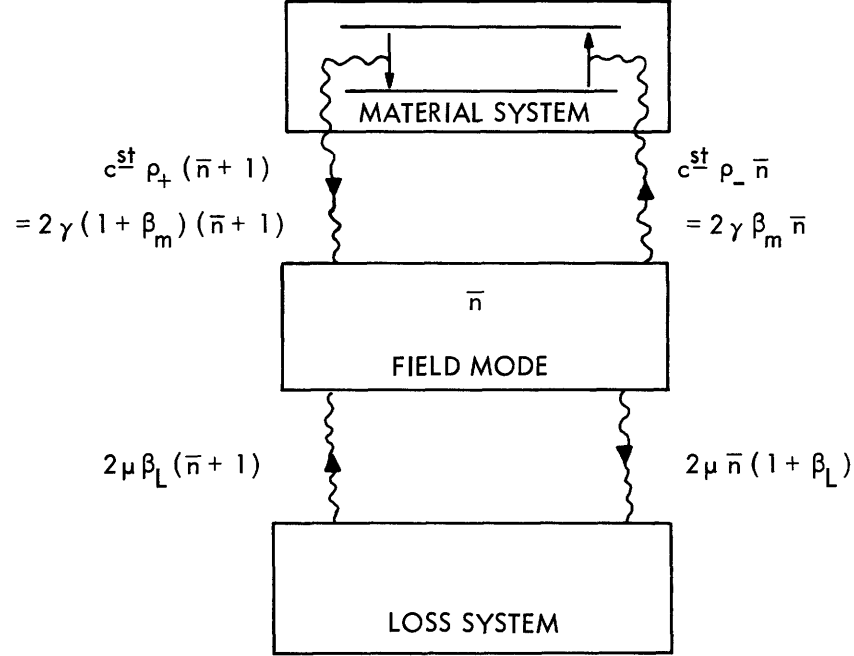


Fig. 6. Rates of photon number transfer if field mode is excited.

Gaussian noise sources. The positive and negative frequency components of the amplitude of this field are denoted by $a_{rv}^+(t)$ and $a_{rv}^-(t)$. They are appropriately normalized (through Eqs. 9) so that the energy of the field is

$$W(t) = \hbar\omega_0 a_{rv}^+(t) a_{rv}^-(t) = \hbar\omega_0 w(t); \quad (122)$$

$w(t)$ is thus the energy of the field in units of $\hbar\omega_0$. The random random variables $a_{rv}^\pm(t)$ commute of course. We assume that the equations of motion of $a_{rv}^\pm(t)$ are given by

$$\frac{da_{rv}^+(t)}{dt} + ga_{rv}^+(t) = x_{rv}^+(t) \quad ; \quad \frac{da_{rv}^-(t)}{dt} + ga_{rv}^-(t) = x_{rv}^-(t), \quad (123)$$

where $x_{rv}^\pm(t)$ are classical random processes the second-order moments of which are given by

$$\langle x_{rv}^+(t+\tau)x_{rv}^-(t) \rangle = \langle x_{rv}^-(t)x_{rv}^+(t+\tau) \rangle = A\delta(\tau) \quad (124a)$$

$$\langle [x_{rv}^+(t)]^r \rangle = \langle [x_{rv}^-(t)]^r \rangle = 0, \quad (r=1, 2). \quad (124b)$$

We assume further that these noise sources are Gaussian, so that all higher order moments are determined from Eqs. 124. The Eqs. 123 are of course suggested by Eqs. 52. We show in Appendix D that they describe a loaded LC circuit driven by noise.

We can solve the Eqs. 123 for the second-order moments in the steady state through the methods of Appendix A. We can obtain higher order moments from the theorem that Gaussian noise sources drive a stable linear system to a Gaussian steady state. We obtain

$$\langle a_{rv}^+(t+\tau)a_{rv}(t) \rangle = \langle a_{rv}(t)a_{rv}^+(t+\tau) \rangle = \frac{A}{2g} e^{-g|\tau|} = G_1(\tau) \quad (125a)$$

$$\langle a_{rv}^+(t+\tau)a_{rv}^+(t) \rangle = \langle a_{rv}(t+\tau)a_{rv}(t) \rangle = 0 \quad (125b)$$

$$\begin{aligned} \langle w(t+\tau)w(t) \rangle &= \langle a_{rv}^+(t+\tau)a_{rv}(t+\tau)a_{rv}^+(t)a_{rv}(t) \rangle \\ &= \left(\frac{A}{2g}\right)^2 + \left(\frac{A}{2g}\right)^2 e^{-2g|\tau|} = G_2(\tau). \end{aligned} \quad (125c)$$

We can also apply previous techniques and find

$$\frac{dw(t)}{dt} + 2gw(t) = x_{rv}^+(t) a_{rv}(t) + x_{rv}^-(t) a_{rv}^+(t) = A + x_{rv}(t) \quad (126a)$$

$$\langle x_{rv}(t+\tau)x_{rv}(t) \rangle = (B-2A) \delta(\tau) \quad ; \quad \langle x_{rv}(t) \rangle = 0, \quad (126b)$$

which leads to

$$\langle w(t) \rangle = G_1(0) = \langle n(t) \rangle, \quad (127a)$$

$$\langle w(t+\tau)w(t) \rangle = G_2(\tau) \quad (127b)$$

Since $a_{rv}^+(t)$ and $a_{rv}(t)$ are Gaussian (in the classical sense), since $a^+(t)$ and $a(t)$ are Gaussian (in the quantum sense), and since the Gaussian moment expansion of a normally-ordered product involves only normally-ordered second-order moments, we can in fact conclude from Eq. 125a that all higher-order moments of $w(t)$ are equal to the corresponding G-functions. In particular, at a single time,

$$\langle [w(t)]^r \rangle = r! \langle w(t) \rangle^r = G_r(0), (r=1, 2, \dots); p_{w(t)}(w) = \frac{1}{\langle w(t) \rangle} e^{-\frac{w}{\langle w(t) \rangle}} \quad (128)$$

We conclude that the equivalent model in which the system variables describe a classical field, and which is described by the Eqs. 123, 124 and the Gaussian character of the noise sources, leads to the correct results for all the G-functions. The constant A has been interpreted in Section 4.8.1: it is the total rate of spontaneous photon emission into the field mode from both the material system and the loss system, or (Eq. 126a) the total rate of energy transfer (in units of $\hbar\omega_0$) into the field mode if the field mode is

unexcited. The function $A\delta(\tau)$ is the second-order moment of the familiar Nyquist source, adapted to quantum statistics and to (linear) loss and gain reservoirs. The Gaussian character of this noise source is also a familiar assumption. The coefficient B has been explained (section 4.8.1) as the total rate of photon emission plus absorption by the material system and loss system. Note, however, that $B-2A$ of Eq. 126b cannot be explained in that way.

4.8.3 Model 2: Classical Particle Model

We shall show that the results for $G_1(0) = \langle n(t) \rangle$ and $G_2(\tau) = \langle n(t+\tau)n(t) \rangle$ can be derived from an equivalent classical model in which the system variable is a number of classical particles (each having a quantum of energy, $\hbar\omega_0$). Let $n_{rv}(t)$ be the number of these particles in the laser cavity. We assume that this number obeys the equation of motion

$$\frac{dn_{rv}(t)}{dt} + 2gn_{rv}(t) = A + x'_{rv}(t); \langle x'_{rv}(t) \rangle = 0 \quad (129)$$

which is suggested by Eq. 113. The interpretation of A was given (section 4.8.1) as the total spontaneous photon emission rate into the field mode. Since $n_{rv}(t)$ is always an integer, it can only change with jumps of magnitude 1. The equation (129) without the noise source can therefore not possibly be correct; it would predict continuous changes in $n_{rv}(t)$. The noise source $x'_{rv}(t)$ must be present and it must be a shot-noise source. From classical random theory we know that the second-order moment of a shot-noise source is the product of the Dirac delta function $\delta(\tau)$ multiplied by the sum of the independent rates, multiplied by the square of the jump ($1^2 = 1$). From Eq. 129 we can read off the net average rate of particle emission into the cavity ($A - 2g\langle n_{rv}(t) \rangle$), but not the total average rate. We assume now that the total average rate is B . The physical interpretation of that assumption has been given (section 4.8.1). We thus have

$$\langle x'_{rv}(t+\tau)x'_{rv}(t) \rangle = B(1^2) \delta(\tau) = B\delta(\tau). \quad (130)$$

From the Eqs. 129 we derive then immediately, in the steady state:

$$\langle n_{rv}(t) \rangle = \frac{A}{2g} = \langle n(t) \rangle = G_1(0) \quad (131a)$$

$$\langle n_{rv}(t+\tau)n_{rv}(t) \rangle = \left(\frac{A}{2g}\right)^2 + \frac{B}{4g} e^{-2g|\tau|} = \langle n(t+\tau)n(t) \rangle = G_2^1(\tau) \neq G_2(\tau). \quad (131b)$$

Higher order moments of the photon number operator n can easily be found by exploiting the Gaussian quantum techniques. We expect that all of these moments can be derived from the previous model, provided we say that the emission and absorption processes are (compound) Poisson processes, that is, each emission and each absorption is an independent event, occurring at rates that are themselves random variables:

$$R_{in}(t) = 2\gamma(1+\beta_m)[n_{rv}(t)+1] + 2\mu\beta_L[n_{rv}(t)-1]$$

$$R_{out}(t) = 2\gamma\beta_m n_{rv}(t) + 2\mu(1+\beta_L) n_{rv}(t)$$

$$B_{rv}(t) = R_{in}(t) + R_{out}(t).$$

For the first-order moment in the steady state, we needed only $A - 2g \langle n(t) \rangle = \langle R_{in}(t) - R_{out}(t) \rangle$; for the second-order moment, only $B = \langle B_{rv}(t) \rangle$. To solve this model for higher order moments is a complicated classical problem; we did not solve it.

We conclude that the moments of the photon number operator can be derived from an equivalent classical model in which the system variable is a number of particles in the laser cavity (each having a quantum of energy $\hbar\omega_0$); the particles are independently emitted and injected one by one. The injection rate is interpreted as stimulated plus spontaneous photon emission into the field mode from both the material and the loss system; the emission rate is interpreted as stimulated photon absorption by the material plus stimulated and spontaneous photon emission into the loss system.

4.9 CONCLUSIONS

We have been able to derive all G_n -functions (normally ordered products) from a classical model in which the system variables describe a classical field, and all G'_n -functions (moments of the photon number operator) from a classical model in which the system variable is a number of classical particles in the cavity. Insight into the internal quantum processes of the laser is required to interpret the moments of the noise sources in the first model, and the emission and injection rates in the second model. If one is willing, however, to accept these moments or these rates, that is, if one is willing to forego an explanation of the internal processes in the laser source, then it is impossible to tell from measurements of the G_n -functions alone that the electromagnetic field produced by the laser oscillator below threshold is anything else than a classical field, and from measurements of G'_n -functions alone, it is impossible to tell that that field is anything else than a number of particles. If one measures both the G_n and the G'_n functions, then one is obliged to accept the fact that that field is a quantum field.

This difference between G_n and G'_n will be called henceforth a pure quantum effect. Unfortunately a device to measure the G'_n -functions has not been described in published works. A photon detector measured the G_n -functions.^{5,6}

V. LASER OSCILLATOR ABOVE THRESHOLD: MODEL 1

5.1 INTRODUCTION

We shall try to find and to solve an operator noise-source formalism for the laser oscillator above threshold. We have explained in Section I why we expect that such a formulation should exist: Haus³ assumed an equivalent classical model for the laser with which he obtained results that were experimentally verified within 10%.^{4, 7} His equivalent model was constructed as follows: He considered the semiclassical field equation for a laser with a collision-broadened linewidth that was much broader than the cold-cavity bandwidth (Van der Pol equation). He inserted in this equation the classical noise source that correctly predicts properties of the field below threshold, but he adapted this source to the actual saturated value of the inversion in the steady state (steady-state source). This classical model is still complicated because it is nonlinear, but "sufficiently" above threshold the fluctuations of the field variables are small compared with their average value, and it is therefore possible to linearize the model. The nonlinear transition region between the linear regime below threshold with its relatively large fluctuations (Gaussian) and the nonlinear regime above threshold with relatively small fluctuations is then excluded. We shall refer to this theory as the "S.L." theory, that, is the semiclassical linearized theory.

In this section we shall restrict ourselves to essentially the same laser as that discussed in the S.L. theory; it will differ only by a somewhat more restricted model for the randomization and pumping mechanisms.

In section 5.2 a detailed description of our laser model is presented. In section 5.3 we set up the exact equations that describe this model. In section 5.4 we transform these equations into the operator noise-source formalism. This leads to a Van der Pol equation for operator variables which contains operator noise sources. We linearize these equations, which again restricts us to operation "sufficiently" above threshold. The approximations that are made are justified in the theses on which this report is based. In this respect, it must be understood that an operator or an operator equation is merely a mathematical tool needed to obtain moments or moment equations for a specific ensemble. An approximation in an operator equation is thus justified if and only if its effect on the resulting moment equations is negligible for that specific ensemble. In section 5.5 we solve the linearized equations and compare the results for second- and fourth-order moments with the results of the S.L. theory.

Our results contain small corrections to the results of the S.L. theory. If the desired accuracy allows us to neglect these corrections, then our theory actually proves the equivalence of the classical model in the S.L. theory.

We summarize the important parameters in terms of which we shall express our results and formulate the restrictions of our model:

$$2\mu = \Delta\omega_0 = \omega_0/Q_0 \quad ; \quad 2\gamma = \omega_0/|Q_m^0| \quad ; \quad 2(\gamma-\mu) = \Delta\omega' = \omega_0/Q'$$

$$\beta_m = \frac{\rho_-}{\rho_+ - \rho_-} \quad ; \quad \beta_m^s = \frac{\rho_-^s}{\rho_+^s - \rho_-^s} \quad ; \quad \rho_+^s - \rho_-^s = \frac{\mu}{\gamma}(\rho_+ - \rho_-) \quad ; \quad \beta_L \quad ;$$

$$\bar{p} = 2\mu\hbar\omega_0 R_0^2 \quad ; \quad 2a\gamma R_0^2 = 2(\gamma-\mu) \quad ; \quad T.$$

The parameter ω_0 is the resonance angular frequency of the cavity mode. The parameter μ is the attenuation constant of the field caused by its coupling to the loss system; the "cold-cavity" bandwidth is 2μ or $\Delta\omega_0$, and the "cold-cavity" quality factor is Q_0 . The parameter γ is the amplification constant of the field caused by its coupling to the inverted material when there is no saturation effect; $|Q_m^0|$ is called the (unsaturated) quality factor of the inverted material. We shall see that $\Delta\omega'$ is the bandwidth of the amplitude fluctuations: it is called the "hot-cavity" bandwidth; Q' is called the "hot-cavity" quality factor. The population of the upper and lower levels of the material, as established by the pump and randomization mechanism alone, are, respectively, ρ_+ and ρ_- ; the actual saturated values of these populations in the steady state, as established by the pump, randomization, and field, are ρ_+^s and ρ_-^s . The relation between saturated and unsaturated values in the steady state is given above. The parameters β_m and β_m^s are measures for the unsaturated and saturated degree of inversion; the parameter β_L measures the temperature, T_L , of the loss system (Eq. 141). The parameter \bar{p} is the average power transmitted in the laser beam (that is, absorbed by the loss system; the average number of photons in the cavity is R_0^2 ; the coefficient a determines the nonlinear saturation effect. T is the average time between collisions, and $1/T$ is the collision bandwidth. Typical order of magnitudes of some of these quantities for the laser used in the Freed and Haus experiments^{4, 11} are $\gamma \simeq \mu \simeq 10^5 \text{ sec}^{-1}$, $1/T = 10^7 \text{ sec}^{-1}$, $R_0^2 = 10^6$ for $(\gamma-\mu) = 10^2 \text{ sec}^{-1}$.

5.2 LASER MODEL

We adopt the following model for the three interacting systems that are essential for any laser model: the field system, the material system, and the loss system (Fig. 7). The field system consist of a single mode of the laser cavity, with resonance angular frequency, ω_0 . This field mode is interacting with N two-level systems ("particles") of the material system. The resonance frequencies of these particles are all equal to the resonance frequency of the mode. Each particle is considered to have a fixed position in the laser cavity, that is, Doppler effects are neglected. We restrict ourselves here to the following model for pumping and randomization of the two-level systems. A particle, j , interacts coherently with the field for some time, t_j . This particle is then taken out of the system and immediately replaced by a new particle. This operation of replacing a particle (henceforth called a collision) occurs so suddenly that no other variable of the

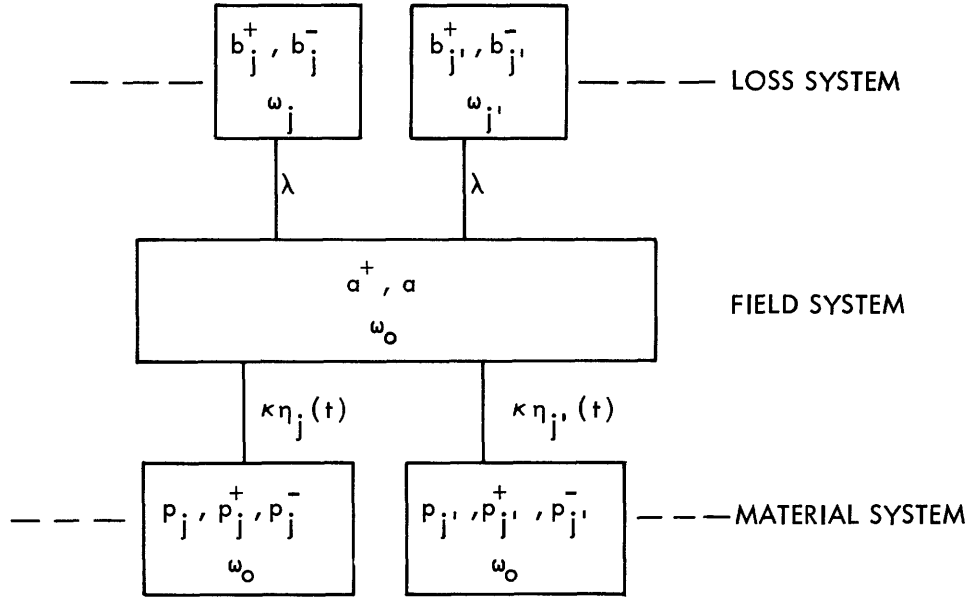


Fig. 7. Interacting systems for laser model 1.

system has time to change (except for its natural time dependence). The state of the new particle immediately after its introduction is characterized by a probability ρ_+ to be in the upper state, and a probability ρ_- to be in the lower state. The time t_j is itself a random variable with an exponential distribution function with mean time T . We impose an essential restriction on our model: the inverse cold-cavity bandwidth must be much larger than T ; in qualitative terms, this means that the field variables do change only infinitesimally in a time T . The collision model of replacing an old particle by a new one is equivalent to the statement that a single particle loses all memory through a collision, in the sense that all possible cross moments of particle operators immediately after the collisions with system operators before the collision (which appear in the final moment equations) have the independence property. The loss system consist of an infinite set of harmonic oscillators, originally in thermal equilibrium and with a frequency-independent distribution of resonant frequencies. The coupling constant to the field mode is the same for all harmonic oscillators.

5.3 EXACT EQUATIONS

We adopt the Hamiltonian

$$\begin{aligned}
 H = \hbar\omega_0 a^+ a + \sum_c \sum_j \left[\hbar\omega_0 \frac{1}{2} p_j - i\hbar\kappa_j (p_j^+ + p_j^-) (a - a^+) \eta_j \right] \\
 + \sum_j \left[\hbar\omega_j b_j^+ b_j^- + \hbar\lambda_j (b_j^+ a + b_j^- a^+) \right].
 \end{aligned} \tag{132}$$

Here, a^+ and a are the creation and annihilation operators of the field mode. The creation and annihilation operators b_j^+ , b_j^- describe the j^{th} harmonic oscillator of the loss system. Its resonant frequency is ω_j . (We introduce $\delta_j = \omega_j - \omega_0$.) λ_j is its coupling constant to the field and is supposed to be independent of j . The operators p_j, p_j^\pm refer to the two-level system j : p_j is the net population operator (upper level-lower level), p_j^+ and p_j^- are proportional to the positive and negative frequency components of its polarization operator; they are also the raising and lowering operators. Their properties were discussed in section 2.3. The function η_j is equal to 1 during a certain interaction time, t_j , and is zero everywhere else. It changes from zero to 1 sufficiently fast so that the system variables cannot change (except for their natural time dependence). The same holds true when η_j changes from 1 to zero. Since there are always N particles interacting with the field, we prefer to use a double summation symbol: \sum_j for the N -interacting particles, \sum_c for the successive collisions. The operators describing the j^{th} particle at the beginning of its interaction time will be denoted $p_j(0), p_j^\pm(0)$. They are independent of and commute with all other system variables at previous times (see previous remark on the equivalence of our collision picture with the statement: A particle loses all memory through a collision). They refer to a "randomized equilibrium" state characterized by a given inversion, that is, by a ρ_+ and ρ_- ; this inversion is assumed to be independent of j . If the particle would not interact with the field it would remain in that state.

After making the rotating-wave approximation, and eliminating the natural time dependence (without changing notation), we obtain from Eq. 132 the following equations of motion:

$$\frac{da(t)}{dt} = \sum_c \sum_j \kappa_j p_j^-(t) \eta_j(t) - \sum_j i\lambda_j b_j^-(t) e^{-i\delta_j t}; \quad \text{and Hermitian conjugate} \quad (133)$$

$$\frac{dp_j^-(t)}{dt} = \kappa_j p_j(t) a(t) \eta_j(t); \quad \text{and Hermitian conjugate} \quad (134)$$

$$\frac{dp_j(t)}{dt} = -2\kappa_j \left[p_j^+(t) a(t) + a^\dagger(t) p_j^-(t) \right] \eta_j(t) \quad (135)$$

$$\frac{db_j^-(t)}{dt} = -i\lambda_j a(t) e^{i\delta_j t}; \quad \text{and Hermitian conjugate.} \quad (136)$$

From Eq. 136 we can derive the integral equation

$$\sum_j i\lambda_j b_j^-(t) e^{-i\delta_j t} = \sum_j i\lambda_j b_j^-(0) e^{-i\delta_j t} + \int_0^t \sum_j \lambda_j^2 e^{-i\delta_j(t-t')} a(t') dt', \quad (137)$$

where $b_j^-(0)$ refers to the original thermal equilibrium ensemble of the j^{th} loss oscillator (the system is launched at $t = 0$). For a flat spectrum and λ_j independent of j , we have

$$\sum_j \lambda_j^2 e^{-i\delta_j(t-t')} = 2\pi\lambda^2 \sigma \delta(t-t') = 2\mu\delta(t-t'), \quad (138)$$

where σ is the density of harmonic oscillators per unit angular frequency, and $\delta(t)$ is the Dirac delta function. Therefore, for $t > +0$,

$$\frac{da(t)}{dt} = \sum_c \sum_j \kappa_j p_j^-(t) \eta_j(t) - \mu a(t) - \sum_j i\lambda_j b_j^-(0) e^{-i\delta_j t}; \quad \text{and Hermitian conjugate} \quad (139)$$

The last term in Eq. 139 is a noise operator, and this form would suffice if we only wanted to treat the linear attenuator. But we shall have to integrate Eq. 139 over an interval, τ . This τ will be one order of magnitude larger than T , and thus much larger than the infinitely small inverse bandwidth of the flat-loss spectrum. We therefore introduce

$$x_L^-(t)\tau = -\sum_j i\lambda_j b_j^-(0) \int_t^{t+\tau} e^{-i\delta_j t'} dt'; \quad \text{and Hermitian conjugate.} \quad (140)$$

Under the assumption of a flat spectrum and by making use of

$$\begin{aligned} \langle (b_j^+)^r (b_j^-)^{r'} \rangle &= \delta_{jj'} \delta_{rr'} r! \beta_L^r; \quad \langle (b_j^-)^r (b_j^+)^{r'} \rangle = \delta_{jj'} \delta_{rr'} r! (1+\beta_L)^r \\ \beta_L &= [\exp(\hbar\omega_0/kT_L) - 1]^{-1} \end{aligned} \quad (141)$$

for a thermal equilibrium ensemble of harmonic oscillators, (see Section IV), one can easily prove that

$$\langle x_L^+(t_1) x_L^-(t_2) \rangle = 2\mu\beta_L(1/\tau) \quad ; \quad \langle x_L^-(t_1) x_L^+(t_2) \rangle = 2\mu(1+\beta_L)(1/\tau) \quad (142a)$$

$$[x_L^-(t_1), x_L^+(t_2)] = 2\mu(1/\tau), \quad (142b)$$

for $t_1 = t_2$; for $|t_1 - t_2| > \tau$, these expressions are zero. x^+, x^- are Gaussian, for example, $\langle x^+(t_1) x^+(t_2) x^-(t_3) x^-(t_4) \rangle = \langle x^+(t_1) x^-(t_3) \rangle \langle x^+(t_2) x^-(t_4) \rangle + \langle x^+(t_1) x^-(t_4) \rangle \langle x^+(t_2) x^-(t_3) \rangle$. Henceforth, Eq. 139 will be written

$$\frac{da(t)}{dt} = \sum_c \sum_j \kappa_j p_j^-(t) \eta_j(t) + \text{Loss}; \quad \text{and Hermitian conjugate} \quad (143)$$

and the loss expressions will be reintroduced when needed. Equations 143, 134, and 135, together with the information we have about $p_j^\pm(0)$, $p_j(0)$ from our collision model, constitute a complete and exact description of our model.

5.4 APPROXIMATIONS

5.4.1 Van der Pol Equations

We subdivide the time in intervals of length τ (Fig. 8). This time duration, τ , is made one order of magnitude larger than T . We consider a particular interaction of duration, t_j , of particle j somewhere in such an interval. The operators describing

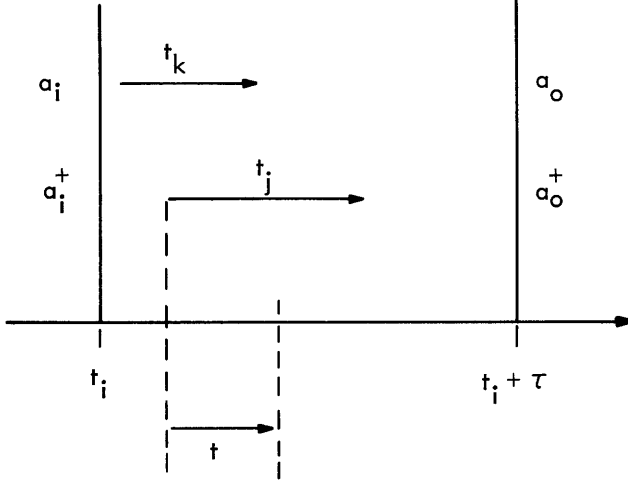


Fig. 8. Interactions of particles j and k in the interval τ .

the field at the beginning (t_i) and the end ($t_i + \tau$) of the interval τ will be denoted a_i, a_i^+ and a_o, a_o^+ , respectively. We shall determine the evolution of particle j during its interaction time, t_j . We could derive it from Eqs. 134 and 135 if the time evolution of $a(t)$ were known. We now make our first approximation: We set $a(t) = a_i$ in these equations during the interval τ . Note that $p_j^\pm(0), p_j(0)$ commute with a_i and a_i^+ , but this is not necessarily so for $p_j^\pm(t), p_j(t)$. We obtain through a series development to order $(\kappa_j t)^3$

$$p_j^-(t) = p_j^-(0) + \kappa_j t p_j(0) a_i + \dots; \quad \text{and Hermitian conjugate} \quad (144)$$

$$p_j(t) = p_j(0) - 2\kappa_j t \left[p_j^+(0) a_i + a_i^+ p_j^-(0) \right] - 2\kappa_j^2 t^2 p_j(0) a_i^+ a_i + \dots \quad (145)$$

Equation 144 has been used to find Eq. 145. Now Eq. 145 can be used to complete $p_j^-(t)$ and Hermitian conjugate:

$$p_j^-(t) = p_j^-(0) + \kappa_j t p_j(0) a_i - \kappa_j^2 t^2 \left[p_j^+(0) a_i^2 + p_j^-(0) a_i^+ a_i \right] - (2/3) \kappa_j^3 t^3 p_j(0) a_i^+ a_i^2 + \dots; \\ \text{and Hermitian conjugate.} \quad (146)$$

We integrate Eq. 143 in the interval $t_i, t_i + \tau$ by means of Eq. 146. We shall drop the argument (0) of $p_j^\pm(0), p_j(0)$; \sum_c will now mean the summation over the collisions in the interval $t_i, t_i + \tau$. We obtain

$$a_o = a_i + \sum_c \sum_j \left[\kappa_j t_j p_j^- + \frac{1}{2} \kappa_j^2 t_j^2 p_j a_i - (1/3) \kappa_j^3 t_j^3 (p_j^+ a_i^2 + p_j^- a_i^+ a_i) - (1/6) \kappa_j^4 t_j^4 p_j a_i^+ a_i^2 \right] \quad (147)$$

$$+ \int_{\tau} \text{Loss} + \dots; \quad \text{and Hermitian conjugate.}$$

We rewrite Eq. 147

$$a_o = a_i + (\gamma - \mu - a \gamma a_i^+ a_i) a_i \tau + x^-(t_i) \tau \quad (148)$$

$$a_o^+ = a_i^+ + a_i^+ (\gamma - \mu - a \gamma a_i^+ a_i) \tau + x^+(t_i) \tau \quad (149)$$

with

$$\gamma \tau = \frac{1}{2} \sum_c \sum_j \kappa_j^2 t_j^2 p_j, \quad \text{and } \langle \gamma \rangle = N \kappa^2 T \langle p \rangle$$

$$\mu = \pi \lambda^2 \sigma \quad (\text{see Eq. 138})$$

$$a \gamma \tau = \sum_c \sum_j (1/6) \kappa_j^4 t_j^4 p_j, \quad \text{and } \langle a \gamma \rangle = 4 N \kappa^4 T^3 \langle p \rangle$$

$$x^-(t_i) = x_L^-(t_i) + x_m^-(t_i), \quad \text{and Hermitian conjugate}$$

$$x_L^-, x_L^+ \quad (\text{see Eqs. 140 and 142})$$

$$x_m^-(t_i) \tau = \sum_c \sum_j \left[\kappa_j t_j p_j^- - (1/3) \kappa_j^3 t_j^3 (p_j^+ a_i^2 + p_j^- a_i^+ a_i) \right]$$

$$x_m^+(t_i) \tau = \sum_c \sum_j \left[\kappa_j t_j p_j^+ - (1/3) \kappa_j^3 t_j^3 (p_j^+ a_i^+ a_i + p_j^- a_i^{+2}) \right]. \quad (150)$$

For later purposes we rewrite the expressions for x_m^-, x_m^+ as

$$x_m^-(t_i) = X_m^-(t_i) + \delta X_m^-(t_i); \quad \text{and Hermitian conjugate}$$

$$X_m^-(t_i) \tau = \sum_c \sum_j \left[\kappa_j t_j p_j^- - (1/6) \kappa_j^3 t_j^3 (p_j^+ a_i^2 + p_j^- a_i^+ a_i) \right]; \quad \text{and Hermitian conjugate}$$

$$\delta X_m^-(t_i) \tau = \sum_c \sum_j \left[- (1/6) \kappa_j^3 t_j^3 (p_j^+ a_i^2 + p_j^- a_i^+ a_i) \right]; \quad \text{and Hermitian conjugate.} \quad (151)$$

The operators p_j^\pm, p_j are independent of $p_{j'}^\pm, p_{j'}$, if $j \neq j'$, in the ensemble to which they refer. The expectation value of p_j^\pm and p_j is zero. Their commutator properties and second-order moments are given in Eqs. 152.

$$\begin{aligned} [p_j^+, p_{j'}^-] &= p_j \delta_{jj'}, \quad ; \quad [p_j, p_{j'}^+] = 2p_j^+ \delta_{jj'}, \quad ; \quad [p_j, p_{j'}^-] = 2p_j^- \delta_{jj'}, \quad ; \\ \langle p_j^+ p_j^- \rangle &= \langle p_j \rangle (1 + \beta_m) \quad ; \quad \langle p_j^- p_j^+ \rangle = \langle p_j \rangle \beta_m \quad ; \end{aligned} \quad (152a)$$

$$\langle p_j \rangle = \rho_+ - \rho_- \quad ; \quad \beta_m = [(\rho_+ / \rho_-) - 1]^{-1} \quad (152b)$$

Equations 148 and 149 are no longer exact for two reasons. First, terms of order $(\kappa_j t_j)^5$ have been neglected. It can be shown that this approximation has a negligible effect on the moments that we want to calculate if $\langle a \gamma a^+ a \rangle$ is much smaller than $\langle \gamma \rangle$, that is, if we do not operate too far above threshold. Second, we have replaced $a(t)$ in Eqs. 134 and 135 by a_i . By such a procedure, we have neglected certain third- and fourth-order terms. The third-order terms do not contain the operators a_i or a_i^+ . The fourth-order terms contain these operators only to the first power. It can be shown that neglecting these terms has a negligible effect on the moments that we want to calculate if $\langle \gamma \rangle T \ll 1$. Above threshold $\langle \gamma \rangle \approx \mu$, so that this restriction means that the inverse cold-cavity bandwidth must be much larger than T .

Equations 148 and 149 are, nevertheless, quantum-mechanically consistent in the sense that they conserve the field commutator. Using Eqs. 150, 142b, and 152a, making approximations consistent with our previous approximations, that is, retaining only terms to order $(\kappa_j t_j)^4$, and neglecting $(\gamma - \mu)^2 \tau^2$ terms, we find from Eqs. 148 and 149

$$\begin{aligned} [a_o, a_o^+] &= 1 + 2\gamma\tau - 2\mu\tau - 4a\gamma\tau a_i^+ a_i - 2\gamma\tau + 2\mu\tau + 4a\gamma\tau a_i^+ a_i \\ &+ \sum_c \sum_j \kappa_j^3 t_j^3 (p_j^+ a_i + p_j^- a_i^+ - \frac{2}{3} p_j^+ a_i - \frac{1}{3} p_j^- a_i^+ - \frac{2}{3} p_j^- a_i^+ - \frac{1}{3} p_j^+ a_i) \\ &= 1. \end{aligned} \quad (153)$$

We first remark that in Eq. 153 the commutator of a_o and a_o^+ is evaluated, and thus not only its expectation value. Second, the terms under the $\sum \sum$ symbol result from the facts that γ is an operator and the field operators $a_i^2, a_i^+ a_i, a_i^{+2}$ appearing in the third-order terms of x_m^\pm do not commute with a_i and a_i^+ . If we approximate $a_i^2, a_i^+ a_i, a_i^{+2}$ in these third-order terms by variables that commute with a_i^+ and a_i , we must consistently consider γ as a c-number in order to preserve the commutator $[a_o, a_o^+]$. Because of the large number of particles and collisions in the interval τ , this c-number would obviously be $\langle \gamma \rangle$.

5.4.2 Linearization Procedure

If we consider τ as a differential dt , Eqs. 148 and 149 can be rewritten

$$\frac{da}{dt} - (\gamma - \mu - \alpha \gamma a^+ a) a = x^-(t) \quad (154)$$

$$\frac{da^+}{dt} - a^+ (\gamma - \mu - \alpha \gamma a^+ a) = x^+(t). \quad (155)$$

These equations are valid if $\langle a \gamma a^+ a \rangle \ll \langle \gamma \rangle$ and $\langle \gamma \rangle T \ll 1$. They can be used (a) "sufficiently" below threshold where the third- and fourth-order terms can be neglected so that we obtain linear equations that can easily be solved with the techniques of Appendix A; (b) through threshold where the nonlinear terms are essential and cannot be linearized; (c) sufficiently above threshold where the nonlinear terms are essential but can be linearized. In discussing this linearization we shall restrict ourselves to operation "sufficiently" above threshold. In Appendix B "sufficiently" is discussed on a quantitative basis.

We introduce the substitution

$$a(t) = [R_0 + \Delta(t)] e^{-i\theta(t)} \quad ; \quad a^+(t) = [R_0 + \Delta^+(t)] e^{i\theta(t)}, \quad (156)$$

in which we postulate that R_0 is a c-number, and $\theta(t)$ a Hermitian operator that commutes with the operators $\Delta(t)$ and $\Delta^+(t)$. Since $[a(t), a^+(t)] = 1$, we must have

$$[\Delta(t), \Delta^+(t)] = 1. \quad (157)$$

There is a unitary transformation relating $a(t)$ and $a^+(t)$ to $\Delta(t)$ and $\Delta^+(t)$. We also postulate that $\theta(t+dt)$ commutes with $\theta(t)$. If we substitute Eq. 156 in Eqs. 154 and 155, we obtain

$$\begin{aligned} -i(R_0 + \Delta) \frac{d\theta}{dt} + \frac{d\Delta}{dt} - \left[\gamma - \mu - \alpha \gamma (R_0^2 + R_0 \Delta^+ + R_0 \Delta + \Delta^+ \Delta) \right] (R_0 + \Delta) \\ = x^-(t) e^{i\theta(t)}; \quad \text{and Hermitian conjugate.} \end{aligned} \quad (158)$$

Thus far, we have only written our equations in another form. We now assume that we can find appropriate variables R_0 and $\theta(t)$ so that all moments formed from Δ and Δ^+ are small compared with the analogous moments of R_0 . We shall then show that Eq. 158 leads to approximate solutions that are consistent with this assumption and with the postulate that R_0 and $\theta(t)$ commute with $\Delta(t)$, $\Delta^+(t)$, and $\theta(t+dt)$. First, this assumption allows us to replace the field variables a_i^2 , $a_i^+ a_i$, a_i^{+2} in the third-order terms of $x_m^\pm(t)$ by $R_0^2 \exp(-2i\theta(t))$, R_0^2 , and $R_0^2 \exp(2i\theta(t))$. The third-order terms of x_m^\pm are indeed responsible for small saturation corrections to the S. L. theory. Making small errors in these third-order terms obviously leads to errors that are orders of magnitude smaller than these saturation corrections. Since the variables $R_0^2 \exp(-2i\theta(t))$, R_0^2 , and $R_0^2 \exp(2i\theta(t))$ commute with $a(t)$ and $a^+(t)$, we must consistently consider γ as the c-number $\langle \gamma \rangle$. We shall use henceforth the notation γ for this c-number. We define R_0^2 by

$$\gamma - \mu - \alpha \gamma R_O^2 = 0, \quad (159)$$

that is, R_O^2 is the number of photons in the laser cavity in the steady state above threshold, as given by a semiclassical analysis. Second, the same assumption allows us to approximate Eq. 158 by

$$-iR_O \frac{d\theta(t)}{dt} + \frac{d\Delta}{dt} + \alpha \gamma R_O^2 (\Delta + \Delta^\dagger) = x_m^-(t) e^{i\theta(t)}; \quad \text{and Hermitian conjugate.} \quad (160)$$

The nonlinear terms $\Delta \frac{d\theta}{dt}$, $\alpha \gamma R_O \Delta^\pm \Delta^\pm$, $\alpha \gamma \Delta^\pm \Delta^\pm \Delta^\pm$ have been neglected. We introduce now the separation of x_m^- (and Hermitian conjugate) into $X_m^- + \delta X_m^-$ (and Hermitian conjugate) that was carried out in Eqs. 151. Equations 160 are equivalent to

$$\begin{aligned} -2iR_O \frac{d\theta(t)}{dt} + \frac{dB(t)}{dt} &= 2i[n_s(t) + \delta n_s(t)] \\ \frac{dA(t)}{dt} + 2\alpha \gamma R_O^2 A(t) &= 2n_c(t), \end{aligned} \quad (161)$$

where A, B, n_s, n_c , and δn_s are defined by

$$\begin{aligned} A &= \Delta + \Delta^\dagger, \quad B = \Delta - \Delta^\dagger \\ 2n_c(t) &= \left[x_L^-(t) + x_m^-(t) \right] e^{i\theta(t)} + \left[x_L^+(t) + x_m^+(t) \right] e^{-i\theta(t)} \\ 2in_s(t) &= \left[x_L^-(t) + X_m^-(t) \right] e^{i\theta(t)} - \left[x_L^+(t) + X_m^+(t) \right] e^{-i\theta(t)} \\ 2i\delta n_s(t) &= \delta X_m^-(t) e^{i\theta(t)} - \delta X_m^+(t) e^{-i\theta(t)}. \end{aligned} \quad (162)$$

From Eqs. 142, 150, 151, and 152, we derive the following properties for $n_c(t)$, $n_s(t)$, and $\delta n_s(t)$: (i) the expectation value of these noise sources is zero: (ii) their second-order moments are given to order $(\kappa_j t_j)^4$ by

$$\begin{aligned} \langle n_s(t+\tau) n_s(t) \rangle &= \left[\gamma \left(\frac{1}{2} + \beta_m \right) + \mu \left(\frac{1}{2} + \beta_L \right) \right] \delta(\tau) = N_s \delta(\tau) \\ \langle n_c(t+\tau) n_c(t) \rangle &= \left[\gamma \left(\frac{1}{2} + \beta_m \right) + \mu \left(\frac{1}{2} + \beta_L \right) - 4\alpha \gamma R_O^2 \left(\frac{1}{2} + \beta_m \right) \right] \delta(\tau) = N_c \delta(\tau) \\ \langle \delta n_s(t+\tau) \delta n_s(t) \rangle &= \langle n_c(t+\tau) n_s(t) \rangle = \langle n_s(t+\tau) \delta n_s(t) \rangle = 0 \\ 2i \langle \delta n_s(t+\tau) n_c(t) \rangle &= i \langle [\delta n_s(t+\tau), n_c(t)] \rangle = \alpha \gamma R_O^2 \delta(\tau); \end{aligned} \quad (163a)$$

(iii) these noise sources are Gaussian. It has indeed been shown in Section IV that the moments of operators that consist of the sum of N operators, independent of each other in a specific ensemble, have the Gaussian property to order $(1/N)$. This ensures the

Gaussian properties of X_m^\pm , δX_m^\pm , and x_m^\pm . The Gaussian properties of x_L^\pm have been mentioned previously. We note that in the steady state N_s can also be written

$$N_s = \mu \left(1 + \beta_m^s + \beta_L \right) = \mu \left(\frac{\rho_+^s}{\rho_+^s - \rho_-^s} + \beta_L \right), \quad (163b)$$

where ρ_+^s and ρ_-^s are the actual saturated values of the population in the steady state:

$$\left(\rho_+^s - \rho_-^s \right) = \left(\rho_+ - \rho_- \right) (\mu / \gamma) \quad ; \quad \beta_m^s = \left[\left(\rho_+^s / \rho_-^s \right) - 1 \right]^{-1}. \quad (163c)$$

5.5 SOLUTION

Equations 161 are for three unknowns, $\theta(t)$, $A(t)$, and $B(t)$. This ambiguity is resolved by setting

$$-2iR_o \frac{d\theta(t)}{dt} = 2in_s(t) \quad (164a)$$

$$\frac{dB(t)}{dt} = 2i\delta n_s(t) \quad (164b)$$

$$\frac{dA(t)}{dt} + 2a\gamma R_o^2 A(t) = 2n_c(t). \quad (164c)$$

In Appendix A a technique is described for solving linear equations driven by noise sources with known moments, for the correlation functions of the unknowns in the steady state. This technique leads to unique solutions under two conditions: (i) there exists a steady state (stable system); and (ii) correlation functions stay finite if the time difference goes to infinity. We introduce the following notation for the steady state:

$$\begin{aligned} \phi_{uv}(\tau) &= \langle u(t+\tau)v(t) \rangle \\ \phi_{[u,v]}(\tau) &= \langle u(t+\tau)v(t) - v(t)u(t+\tau) \rangle. \end{aligned} \quad (165)$$

In the steady state, one has $\phi_{vu}(\tau) = \phi_{uv}(-\tau) + \phi_{[v,u]}(\tau)$, and $\phi_{[v,u]}(\tau) = -\phi_{[u,v]}(-\tau)$. If we apply the techniques of Appendix A to Eqs. 164, we find

$$\phi_{\theta\theta}(\tau) = \phi_{\theta\theta}(0) - \left(N_s / 2R_o^2 \right) |\tau| \quad (166a)$$

$$\phi_{BB}(\tau) = \phi_{BB}(0) = -1$$

$$\phi_{AA}(\tau) = \left(N_c / a\gamma R_o^2 \right) \exp(-2a\gamma R_o^2 |\tau|) \quad (166b)$$

$$\phi_{BA}(\tau) = \eta(\tau) + \eta(-\tau) \exp(-2\alpha\gamma R_0^2 |\tau|)$$

$$\phi_{AB}(\tau) = -\eta(-\tau) - \eta(\tau) \exp(-2\alpha\gamma R_0^2 |\tau|)$$

$$\phi_{[B,A]}(\tau) = -\phi_{[A,B]}(-\tau) = 2\eta(\tau) + 2\eta(-\tau) \exp(-2\alpha\gamma R_0^2 |\tau|), \quad (166c)$$

where $\eta(\tau) = +1$ for $\tau > 0$, and $\eta(\tau) = 0$ for $\tau < 0$. All other second-order moments, cross moments, or expectation values of commutators are zero. The actual value of $\phi_{BB}(0)$ in Eq. 166b is not important; it will only add DC terms to the first-order and second-order Glauber functions. These DC terms have no experimental significance; only the τ -dependence of these Glauber functions is important. We only mention that the value $\phi_{BB}(0) = -1$ corresponds to the existence of a $p(\alpha)$ representation for the density matrix of the field in the steady state (see Appendix A). We showed in Section IV that the classical theorem that stationary Gaussian noise sources drive a stable linear system to a Gaussian steady state is also valid in the quantum case. The variables θ , A , and B are thus Gaussian in the steady state, and the properties of all higher moments formed with θ , A , and B can be derived from Eqs. 166. We repeat the statement of the introduction that an operator property can be said to be "effectively" true if it is true for all moments in a specific ensemble to which one restricts himself. We can now verify the consistency of our solutions with the original assumptions and postulates; (i) $\theta(t+\tau)$ "effectively" commutes with $A(t)$ and $B(t)$ for all τ , and therefore, in particular for $\tau = 0$; (ii) $\theta(t+\tau)$ "effectively" commutes with $\theta(t)$ for all τ and therefore $d\theta(t)/dt$ "effectively" commutes with $\theta(t)$; (iii) all moments of A and B are small with the corresponding moments of R_0 . We can also verify that $\theta(t+\tau)$ is independent of $A(t)$ and $B(t)$.

In view of published remarks^{13,15} that $\langle a^\pm(t) \rangle$ is zero in the steady state, we present the following discussion. In order to describe time evolution from t to a later time $t + \tau$, we can construct a subspace for the field at t and a subspace for the noise sources between t and $t + \tau$. At t , a and a^\dagger are identity matrices in the noise-source subspace, and θ is an identity matrix in the field subspace. As time evolves, a , a^\dagger , and θ develop components in the noise-source subspace, but θ remains an identity matrix in the field subspace. For the calculation of field moments at t only, we can treat $\theta(t)$ as a c-number and therefore we can visualize the field ensemble at t as consisting of a classical ensemble of subensembles. For each subensemble, $\theta(t)$ is a deterministic number. $\theta(t)$ might be a classical random variable for the classical ensemble of subensembles. We could prepare the ensemble at t so that $\theta(t)$ is a deterministic number for the total ensemble. Equation 164a can still be solved and leads to $\langle \theta(t+\tau) \rangle = \theta(t)$ and $\langle [\theta(t+\tau) - \theta(t)]^2 \rangle = \langle [\theta(t+\tau) - \langle \theta(t+\tau) \rangle]^2 \rangle = (N_s/R_0^2)\tau$. This shows that at $t+\tau$, $\theta(t+\tau)$ is no longer deterministic. Clearly, then, this ensemble is not the steady-state ensemble. If, however, we choose the initial ensemble such that $\theta(t)$ is a Gaussian variable with an infinitely large spread, then $\theta(t+\tau) = \theta(t) + \theta(t+\tau) - \theta(t)$ is also a Gaussian variable with

the same spread. This new ensemble is the steady-state ensemble, and $\langle e^{\pm i\theta(t)} \rangle = 0$; therefore, $\langle a(t) \rangle = \langle a^+(t) \rangle = 0$.

We use these properties of θ , A , and B to calculate the field moments G_1 , G_2 , $\langle [a(t+\tau), a^+(t)] \rangle$, and G_2' , where

$$\begin{aligned} G_1 &= \langle a^+(t+\tau)a(t) \rangle \\ &= \langle e^{i[\theta(t+\tau)-\theta(t)]} \rangle \left[R_o^2 + R_o(\langle \Delta^\pm \rangle) + \langle \Delta_{t+\tau}^+ \Delta_t \rangle \right] \\ &= \left[R_o^2 + \phi_{\Delta^+ \Delta}(\tau) \right] \exp[\phi_{\theta\theta}(\tau) - \phi_{\theta\theta}(0)] \end{aligned} \quad (167)$$

$$\begin{aligned} G_2 &= \langle T^+(a^+(t)a^+(t+\tau))T(a(t+\tau)a(t)) \rangle \\ &= R_o^4 + R_o^3(\langle \Delta^\pm \rangle) + R_o^2 \left[\langle (\Delta_t^+ \Delta_{t+\tau}^+ + \Delta_t^+ \Delta_{t+\tau} + \Delta_t^+ \Delta_t^+ + \Delta_{t+\tau}^+ \Delta_{t+\tau} + \Delta_{t+\tau}^+ \Delta_t^+ + \Delta_{t+\tau}^+ \Delta_t) \rangle \right] \\ &\quad + R_o(\langle \Delta^\pm \Delta^\pm \Delta^\pm \rangle) + \langle \Delta_t^+ \Delta_{t+\tau}^+ \Delta_{t+\tau} \Delta_t \rangle \\ &= R_o^4 + R_o^2 \left[\phi_{AA}(\tau) - \phi_{[A, \Delta^+]}(\tau) + 2\phi_{\Delta^+ \Delta}(0) \right], \quad (\tau > 0) \end{aligned} \quad (168)$$

$$\langle [a(t+\tau), a^+(t)] \rangle = \phi_{[\Delta, \Delta^+]}(\tau) \exp[\phi_{\theta\theta}(\tau) - \phi_{\theta\theta}(0)], \quad (169)$$

$$\begin{aligned} G_2' &= \langle a^+(t+\tau)a(t+\tau)a^+(t)a(t) \rangle \\ &= R_o^4 + R_o^2 \left[\phi_{AA}(\tau) + 2\phi_{\Delta^+ \Delta}(0) \right]. \end{aligned} \quad (170)$$

In Eqs. 167 and 169 we have made use of the fact that θ is a Gaussian variable independent of A and B . In the left-hand side of Eq. 168 we introduced the time-ordering operators T (which puts the later time first) and T^+ (which puts the earlier time first). These operators make G_2 a symmetric function of τ in the steady state, even when $[a(t+\tau), a(t)]$ and $[a^+(t+\tau), a^+(t)]$ are not zero. In the steady state below threshold these commutators are zero because a is not coupled to a^+ . In the steady state above threshold a is coupled to a^+ and one must not expect these commutators to be zero. Indeed, they are not zero, as can be checked, for example, from $\langle a^+(t)a^+(t+\tau)[a(t+\tau), a(t)] \rangle = R_o^2 \phi_{[\Delta, \Delta^+]}(\tau)$. The operators T and T^+ are not needed in Glauber's definition⁵ of G_2 because he deals with fields sufficiently far from the source, which presumably have the free-field time propagation so that $[E^-(t), E^-(t')] = 0$ and $[E^+(t), E^+(t')] = 0$. We deal here with one mode inside the laser cavity. Since our G_2 is symmetric in (τ) , we have calculated in the right-hand side of Eq. 168 only its expression for $\tau > 0$. Furthermore, we have neglected the contribution of the fourth-order moment of Δ^\pm . It can be shown (see Appendix B) that through

linearization we have made errors in the second-order moments which when multiplied with R_o^2 are of the same order as this fourth-order moment. An unfortunate but consistent consequence of the linearization approximation is that the higher order moments of the field do not teach us much more than the lower order moments because they are both expressed in terms of the same order of moments of Δ^\pm . The function G_2' in Eq. 170 was only calculated to show that its τ -dependence can be expressed in terms of $\phi_{AA}(\tau)$ alone.

Since $\Delta = (A+B)/2$ and $\Delta^+ = (A-B)/2$, we can calculate the moments in Eqs. 167-170 from Eqs. 166, and we find

$$G_1 = \left[R_o^2 + \frac{1}{4} \left(\frac{N_c}{\alpha \gamma R_o^2} - 1 \right) e^{-2\alpha \gamma R_o^2 |\tau|} \right] e^{-(N_s/2R_o^2) |\tau|} \quad (171)$$

$$\frac{G_2 - R_o^4}{R_o^2} = \left(\frac{N_c}{\alpha \gamma R_o^2} - 1 \right) \left(e^{-2\alpha \gamma R_o^2 |\tau|} + \frac{1}{2} \right) \quad (172)$$

$$\langle [a(t+\tau), a^\dagger(t)] \rangle = \frac{1}{2} \left(1 + e^{-2\alpha \gamma R_o^2 |\tau|} \right) e^{-(N_s/2R_o^2) |\tau|}. \quad (173)$$

The τ -dependence of $\langle [a(t+\tau), a^\dagger(t)] \rangle$ is shown in Fig. 9. For small values of τ , it decays with the time constant of the amplitude fluctuations ($1/2\alpha \gamma R_o^2$); for large values of τ , it

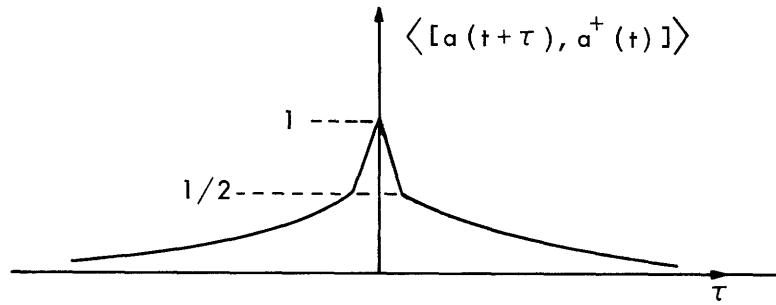


Fig. 9. Expectation value of the commutator $[a(t+\tau), a^\dagger(t)]$ versus τ .

decays with the much larger time constant of the phase diffusion ($2R_o^2/N_s$). The main term of G_1 is $R_o^2 \exp[-N_s |\tau|/2R_o^2]$. The spectrum of this term is a Lorentzian with full half-power width given by

$$\Delta\omega = (N_s/R_o^2) = \frac{\hbar\omega_o}{2p} (\Delta\omega_o)^2 \left[\frac{\rho_+^s}{\rho_+^s - \rho_-^s} + \beta_L \right], \quad (174)$$

where $\Delta\omega_0 = 2\mu$ is the cold-cavity bandwidth, $\bar{p} = 2\mu\hbar\omega_0 R_0^2$ is the power transmitted in the laser beam, β_L is a temperature factor of the loss system defined in Eq. 141, ρ_+^s and ρ_-^s are the actual saturated values of the population of the inverted material; they can be calculated from the unsaturated values through Eq. 163c. The DC term on the right-hand side of (172) has no experimental significance; it can be added to R_0^4 on the left-hand side. We can express (172) in experimental terms as

$$\frac{G_2(\tau) - G_2(\infty)}{R_0^2} = \left[\frac{Q'}{Q_0} \left(\frac{\rho_+^s}{\rho_+^s - \rho_-^s} + \beta_L \right) - (3+4\beta_m) \right] \exp \left[-\frac{\omega_0}{Q'} |\tau| \right], \quad (175)$$

where Q' is the "hot-cavity" quality factor (defined by $2(\gamma-\mu) = \omega_0/Q'$), Q_0 is the "cold-cavity" quality factor (defined by $2\mu = \omega_0/Q_0$), and β_m is a temperature factor for the unsaturated material defined by Eq. 152b. We can also express this result in terms of the Fourier transform $\Phi_p(\omega)$ of $(2\mu\hbar\omega_0)^2 [G_2(\tau) - G_2(\infty)]$ by making use of Eq. 105. We find

$$\frac{\Phi_p(\omega)}{\bar{p}} = \frac{1}{\pi} \frac{\omega_0^2}{\omega^2 + (\omega_0/Q')^2} \frac{\hbar\omega_0}{Q_0^2} \left[\frac{\rho_+^s}{\rho_+^s - \rho_-^s} + \beta_L - \frac{Q_0}{Q'} (3+4\beta_m) \right]. \quad (176)$$

We finally note that the important factor $Q'/Q_0 = \mu/(\gamma-\mu)$ is called the "enhancement" factor.

We shall now discuss the detailed structure of G_1 and G_2 by means of an equivalent classical problem. We start from the Van der Pol equations (154) and (155), but with a and a^+ now random variables, and with other noise sources. We linearize these equations by means of the substitution

$$a(t) = [R_0 + R_1(t)] e^{-i\theta(t)} \quad ; \quad a^+(t) = [R_0 + R_1(t)] e^{i\theta(t)}. \quad (177)$$

This leads to

$$-R_0 \frac{d\theta(t)}{dt} = n'_s; \quad \frac{dR_1(t)}{dt} = 2a\gamma R_0^2 R_1(t) = n'_c. \quad (178)$$

If we postulate that n'_s and n'_c are independent Gaussian noise sources with average value zero, and with a correlation function for n'_s equal to $N_s \delta(\tau)$ and for n'_c to $(N_c - a\gamma R_0^2) \delta(\tau) = [N_s - a\gamma R_0^2 (3+4\beta_m)] \delta(\tau)$, then Eqs. 178 lead to the exact results for G_1 and G_2 . In the S.L. theory with the steady-state source one finds instead that the correlation functions of n'_s and n'_c are equal to $N_s \delta(\tau)$. The S.L. theory with the steady-state source thus predicts correctly the phase-diffusion time constant, but gives rise to an error of $(3+4\beta_m)$ photons in the relative photon-number fluctuations $[G_2 - R_0^4]/R_0^2$. Close to threshold, these $(3+4\beta_m)$ photons are small compared with the main term of order $\mu/(\gamma-\mu)$ photons, but they become relatively more important higher

above threshold because the main term decreases. If $(N_c - \alpha \gamma R_o^2)$ were equal to N_s , the steady-state source would be exact. It is interesting to investigate the cause of the difference between these two expressions. First, N_c is different from N_s ; this can be called a saturation correction because it is caused by the nonlinear contributions to the material noise sources; this is responsible for a correction of $(2+4\beta_m)$ photons in the relative photon-number fluctuations. Second, $N_c - \alpha \gamma R_o^2$ is different from N_c ; this can be called a quantum correction because it is caused by the fact that the variables in Eq. 161 are operators, and is responsible for a correction of 1 photon to the relative photon-number fluctuations. This quantum correction is not present in the expression for G_2' (Eq. 170). Finally, we would expect to find another quantum correction if we calculated the field moments in the laser beam outside the cavity.

5.6 CONCLUSIONS

We have calculated the expectation value of the field commutator and the first- and second-order Glauber functions by means of the operator noise-source formalism for the steady state of the laser oscillator above threshold. We have considered a laser model oscillating in one mode with resonance frequency tuned to the inverted material. Randomization and pumping of the material system has been represented by one type of collision. An essential restriction has been imposed on that model: The collision-broadened linewidth of the material has been assumed to be much larger than the cold-cavity bandwidth ($\mu T \ll 1$).

Our main results are summarized in Eqs. 171-173. If we neglect the (small) influence of the amplitude fluctuations on G_1 , then the Fourier transform of G_1 is Lorentzian with full-half power width given by Eq. 174. This result is identical to the recently announced result of Lax,¹³ if one restricts his result to the conditions of our model: no detuning and $\mu T \ll 1$. It is identical to the result obtained by Haus³ by means of the S. L. theory. The relative photon-number fluctuations and the spectrum of the power fluctuations are given in Eq. 175 and 176. The result (176) should be compared with Equation 20 of Freed and Haus,⁷ which was derived by means of the S. L. theory. They supposed that the loss system was at zero temperature. The last term of the factor between square brackets describes the quantum and saturation corrections of our theory. They are small compared with the other terms because Q_o/Q' is small. Their detailed structure has been discussed.

We have found that the second-order Glauber function is only symmetric in the time-difference because we introduced time-ordering operators. This was related to the fact that the creation (or annihilation) operators at different times did not commute. Glauber dealt with fields far from sources, and he assumed that such fields are free and have thus symmetric G functions.⁵ We dealt here with the field inside the laser cavity. In Section VII, we shall investigate the radiation of the laser into the outside space.

VI. LASER OSCILLATOR: MODEL 2

6.1 INTRODUCTION

We shall now analyze a more general laser model in which no restrictions are put on the relative magnitude of the various relaxation times. We shall give a description of the model and then a detailed analysis of the noise-source formulation of the interaction between the material system and its reservoir. The noise-source formulation of the interaction between the field system and its reservoir (loss system) is taken from the previous sections. These formulations are then combined to obtain the noise-source formulation of the equations of motion of the laser variables. These equations are solved, first, below threshold, and second, in various special cases above threshold. Above threshold we use a linearization approximation in which no attempt is made to retain complete quantum-mechanical consistency, that is, the commutator of certain noise sources will be neglected. It is equivalent to neglecting what we have called "the quantum corrections."

6.2 LASER MODEL

The field system consists of one mode, with resonance angular frequency ω_0 . The material system consists of a set of many-level systems (particles). In each particle j there is one level pair, with resonance angular frequency ω_{0j} , which is coupled to the field mode (coupling constant κ_j). The field system is also coupled to a loss reservoir (see section 2.4) and the noise-source formulation of its interaction with the field system is given in section 3.3. The material system is also coupled to the material reservoir, which causes pumping, nonradiative decay and randomization of the material system. We consider two types of reservoir: the first type causes transitions among the levels; the second type causes only pure phase shifts in the levels. These reservoirs will be described in more detail below.

6.3 NOISE-SOURCE FORMULATION OF THE INTERACTION OF A MANY-LEVEL SYSTEM WITH ITS RESERVOIR

We consider a single many-level system. We analyze its interactions with a transition-inducing reservoir, a pure phase-shift inducing reservoir, and with both reservoirs at the same time.

6.3.1 Transition-Inducing Reservoir

We assume that each level pair (ij) is coupled to its own reservoir $(R(ij))$ consisting of its own set of harmonic oscillators. The various harmonic oscillators of $R(ij)$ are supposed to be distributed continuously, uniformly and symmetrically around the resonance angular frequency $|\omega_{ij}|$ of the level pair (ij) , with density $|\sigma_{ij}|$ per unit angular frequency, over an infinitely wide bandwidth. These harmonic oscillators (distinguished by the superscript ν) are supposed to be independent of each other at launching time $t = 0$;

their initial state is characterized by the density matrix

$$\rho_{R(ij)}^{(0)} = \prod_{\nu} \rho_{R(ij)}^{\nu}(0); \quad \rho_{R(ij)}^{\nu}(0) = \sum_n |n\rangle \langle n| p_{ij}(n). \quad (179a)$$

We set

$$\bar{n}_{ij} = \sum n p_{ij}(n). \quad (179b)$$

In the rotating-wave approximation, we have the following Hamiltonian for the many-level system and its reservoir

$$H = \sum_i \hbar \Omega_i s_{ii} + \sum_{i,j} \sum_{\nu} \left[\frac{1}{2} \hbar |\omega_{ij}^{\nu}| b_{ij}^{\nu} b_{ji}^{\nu} + \hbar s_{ij} h_{ij}^{\nu} b_{ij}^{\nu} \right], \quad (180)$$

where b_{ij}^{ν} is the annihilation operator of the ν^{th} harmonic oscillator of the (ij) reservoir if i is above j , and the creation operator of this harmonic oscillator if i is below j ; when $i = j$ we set $b_{ii}^{\nu} = 0$. We refer to section 2.3 for a description of the variables s_{ij} of the many-level system. The Hermitian character of H requires $h_{ji}^{\nu} = (h_{ij}^{\nu})^*$. When $i = j$ we set $h_{ii} = 0$ and we further assume $|h_{ij}^{\nu}|^2 = |h_{ij}|^2$ is independent of ν . The reader can easily check that the interaction Hamiltonian in Eq. 180 is just a generalization of the interaction Hamiltonian between a level pair and a field mode, which is given in Eq. 20. In Eq. 180 we consider many modes coupled to a level pair; furthermore, we think now of these modes as running waves (e.g., black-body modes): the coupling constant is therefore in general complex. The assumption that each level pair is coupled to its own set of modes is simply a realistic idealization for the case in which the resonance angular frequencies $|\omega_{ij}|$ of the various level pairs have no degeneracies.

We introduce ω_{ij} , ϵ_{ij} , ω_{ij}^{ν} , and δ_{ij}^{ν} defined by

$$\begin{aligned} \omega_{ij} &= \Omega_i - \Omega_j; \\ \epsilon_{ij} &= +1 \text{ if } i \text{ above } j; \quad \epsilon_{ij} = -1 \text{ if } i \text{ below } j; \quad \epsilon_{ij} = 0 \text{ if } i = j; \\ \omega_{ij}^{\nu} &= \epsilon_{ij} |\omega_{ij}^{\nu}|; \quad \delta_{ij}^{\nu} = \omega_{ij}^{\nu} - \omega_{ij}. \end{aligned} \quad (181)$$

If we use the commutator relations among the s_{ij} (implied by Eq. 13) and between the creation and annihilation operator of a harmonic oscillator, we obtain for the equations of motion

$$\frac{db_{ij}^{\nu}}{dt} = -i\omega_{ij}^{\nu} b_{ij}^{\nu} + ih_{ji}^{\nu} s_{ji} \epsilon_{ji} \quad (182a)$$

$$\frac{ds_{ij}}{dt} = i\omega_{ij} s_{ij} - i[s_{ij}, s_{kl}] \sum_{\nu} h_{kl}^{\nu} b_{kl}^{\nu} \quad (182b)$$

We eliminate the natural time dependences by substituting

$$s_{ij} \rightarrow s_{ij}(t) e^{i\omega_{ij}t} ; \quad b_{ij}^\nu \rightarrow b_{ij}^\nu(t) e^{-i\omega_{ij}^\nu t} \quad (183)$$

in Eqs. 182, so that we obtain for the equations of motion of the new slowly time-variant operators

$$\frac{db_{ij}^\nu(t)}{dt} = ih_{ji}^\nu e^{-i\delta_{ji}^\nu t} s_{ji}(t) \epsilon_{ji} \quad (184a)$$

$$\frac{ds_{ij}(t)}{dt} = -i \left[s_{ij}(t), s_{kl}(t) \right] \sum_{\nu} h_{kl}^\nu e^{-i\delta_{kl}^\nu t} b_{kl}^\nu(t). \quad (184b)$$

We introduce

$$f_{kl}(t) = \sum_{\nu} h_{kl}^\nu e^{-i\delta_{kl}^\nu t} b_{kl}^\nu(t) ; \quad x_{kl}(t) = \sum_{\nu} h_{kl}^\nu e^{-i\delta_{kl}^\nu t} b_{kl}^\nu(0) \quad (185)$$

From Eq. 184a we deduce the integral equation

$$b_{kl}^\nu(t) = b_{kl}^\nu(0) + i \int_0^t h_{lk}^\nu e^{-i\delta_{lk}^\nu t'} \epsilon_{lk} s_{lk}(t') dt' \quad (186)$$

so that

$$f_{kl}(t) = x_{kl}(t) + i \int_0^t \sum_{\nu} |h_{lk}|^2 e^{i\delta_{lk}^\nu(t-t')} \epsilon_{lk} s_{lk}(t') dt'. \quad (187)$$

The assumptions that we made about the reservoir allow us to put

$$\sum_{\nu} e^{i\delta_{ij}^\nu t} = 2\pi |\sigma_{ij}| \delta(t) ; \quad \int_{-\epsilon}^0 \delta(t) dt = \int_0^{+\epsilon} \delta(t) dt = \frac{1}{2}, \quad (188)$$

where ϵ is an arbitrary small positive number. We can thus transform (187) to

$$f_{kl}(t) = x_{kl}(t) + i a_{lk} s_{lk}(t) \quad (189)$$

with

$$a_{lk} = \epsilon_{lk} |h_{lk}|^2 \pi |\sigma_{lk}|, \quad (190)$$

and therefore

$$\frac{ds_{ij}(t)}{dt} = -i[s_{ij}(t), s_{kl}(t)] x_{kl}(t) + [s_{ij}(t), s_{kl}(t)] s_{lk}(t) a_{lk} \quad (191)$$

or, using Eq. 13,

$$\frac{ds_{ij}(t)}{dt} = -i[s_{ij}(t), s_{kl}(t)] x_{kl}(t) + [s_{ij}(t) + \delta_{ij} s_{kk}(t)] a_{kj} \quad (192)$$

In Eqs. 182b, 184b, 191 and 192 the Einstein summation notation was used.

Equation 192 is not of the Langevin type, that is, it is not split in a first term (averaged equation) containing only system variables (s) and a second term that is a noise source with zero average value. It is, however, of the type discussed in Appendix C. From (179), (185), and (188) it follows that $x(t)$ fulfills all of the conditions put on $x(t)$ in Appendix C. We have, in particular,

$$\langle x_{kl}(t) x_{mn}(t') \rangle_R = w_{mn} \delta_{kn} \delta_{lm} \delta(t-t') \quad (193)$$

$$w_{mn} = 2 |a_{mn}| \bar{n}_{mn} \quad \text{if } m \text{ above } n$$

$$= 2 a_{mn} (\bar{n}_{mn} + 1) \quad \text{if } m \text{ below } n$$

$$= 0 \quad \text{if } m = n \quad (194)$$

$$w_{mn} + 2a_{mn} = w_{nm} \quad (195)$$

In Appendix C it is shown that equations of the type (192) can be transformed to the Langevin form, the averaged equations can be found from the drift terms (a -terms) in the first-order term of a perturbation expansion and from the "xx" terms in the second-order term of the perturbation expansion, and the second-order moments of the Langevin forces (noise sources) can be found from cross-multiplying the "x" terms of the first-order term of the perturbation expansion. From (192) it follows that the first-order term and the second-order "xx" term of the perturbation expansion are, respectively,

$$\begin{aligned} & \frac{1}{Dt} \left\{ -i \int_t^{t+Dt} [s_{ij}(t), s_{kl}(t)] x_{kl}(t_1) dt_1 + \int_t^{t+Dt} [s_{ij}(t) + \delta_{ij} s_{kk}(t)] a_{kj} dt_1 \right\}, \\ & \frac{1}{Dt} \left\{ (-i)^2 [[s_{ij}(t), s_{kl}(t)], s_{mn}(t)] \int_t^{t+Dt} dt_1 \int_t^{t_1} dt_2 x_{mn}(t_2) x_{kl}(t_1) \right\}. \end{aligned} \quad (196)$$

If now we set

$$\frac{ds_{ij}(t)}{dt} = A_{ij}(t) + F_{ij}(t) \quad ; \quad \langle F_{ij}(t) \rangle = 0, \quad (197)$$

then it follows from Eqs. C. 7, C. 9, 193 and 196 that

$$A_{ij}(t) = -[[s_{ij}(t), s_{kl}(t)], s_{mn}(t)] \frac{1}{2} w_{kl} \delta_{ml} \delta_{nk} + [s_{ij}(t) + \delta_{ij} s_{kk}(t)] a_{kj}. \quad (198)$$

From Eqs. C. 13, 193 and 196 it follows that

$$\begin{aligned} \langle F_{ij}(t) F_{pq}(t') \rangle &= \delta(t-t') \{ \dots \} \\ \{ \dots \} &= -\langle [s_{ij}(t), s_{kl}(t)] [s_{pq}(t), s_{mn}(t)] \rangle_R w_{mn} \delta_{kn} \delta_{lm}. \end{aligned} \quad (199)$$

If we use Eqs. 13 and 195, we can easily transform Eqs. 198 and 199 to

$$A_{ij}(t) = \sum_k \left[-s_{ij}(t) \left(\frac{1}{2} w_{ki} + \frac{1}{2} w_{kj} \right) + \delta_{ij} s_{kk} w_{ik} \right] \quad (200a)$$

$$\begin{aligned} \langle F_{ij}(t) F_{pq}(t') \rangle_R &= \delta(t-t') \left[\sum_k \langle s_{iq}(t) \rangle_R w_{kj} \delta_{jp} + \sum_k \langle s_{kk}(t) \rangle_R w_{ik} \delta_{iq} \delta_{jp} \right. \\ &\quad \left. - \langle s_{ij}(t) \rangle_R w_{pj} \delta_{pq} - \langle s_{pq}(t) \rangle_R w_{ip} \delta_{ij} \right]. \end{aligned} \quad (200b)$$

The interpretation of Eqs. 200 will be given in section 6.3.3.

6.3.2 Pure Phase Shift-Inducing Reservoir

We assume that a reservoir exists that introduces collisionlike perturbations in the many-level system: Each time a collision c of type (i) occurs, it introduces an independent random phase shift $\exp(i\phi_i^c)$ in level $|i\rangle$ with

$$\langle e^{i\phi_i^c} \rangle_R = 0. \quad (201)$$

These collisions occur at a rate γ_i in Poisson-like fashion, that is, the probability of occurring in dt is given by $\gamma_i dt$ and different collisions are independent events. The equation of motion for $s_{ij}(t)$ (slowly time-variant) is

$$\frac{ds_{ij}(t)}{dt} = \sum_{t_c} \left[s_{ij}(t_c^+) - s_{ij}(t) \right] \delta(t - t_c^-), \quad (202)$$

where t_c is the time at which collision c occurs, and t_c^+ and t_c^- are, respectively, the time immediately after and before that collision. For a collision c of type k occurring at time t_c we have

$$s_{ij}(t_c^+, k) = s_{ij}(t_c^-) \left[\delta_{ij} + (1 - \delta_{ij}) \left(e^{i\phi_i^c} \delta_{ik} + e^{-i\phi_j^c} \delta_{jk} \right) \right]. \quad (203)$$

We have

$$\begin{aligned} \left\langle s_{ij}(t_c^+, k) \right\rangle_R &= \left\langle s_{ij}(t_c^-) \right\rangle_R \delta_{ij}, \\ s_{ij}(t_c^+, k) - s_{ij}(t_c^-) &\neq 0 \quad \text{iff } i \neq j \text{ and } k = i \text{ or } j. \end{aligned} \quad (204)$$

From these assumptions and equations we obtain immediately

$$\begin{aligned} \left\langle \frac{ds_{ij}(t)}{dt} \right\rangle_R &= \sum_k \gamma_k \left\langle s_{ij}(t^+, k) - s_{ij}(t) \right\rangle_R \\ &= - \left\langle s_{ij}(t) \right\rangle_R (1 - \delta_{ij})(\gamma_i + \gamma_j) \\ &= - \left\langle s_{ij}(t) \right\rangle_R \Gamma_{ij}^{\text{ph}}. \end{aligned} \quad (205a)$$

Here, we have set

$$\Gamma_{ij}^{\text{ph}} = \Gamma_{ji}^{\text{ph}} = (1 - \delta_{ij})(\gamma_i + \gamma_j). \quad (205b)$$

If we set

$$\frac{ds_{ij}(t)}{dt} = A_{ij}(t) + F_{ij}(t) \quad (206)$$

with

$$A_{ij}(t) = -s_{ij}(t) \Gamma_{ij}^{\text{ph}}; \quad (207a)$$

$$F_{ij}(t) = \sum_{t_c} \left[s_{ij}(t_c^+) - s_{ij}(t) \right] \delta(t - t_c^-) - A_{ij}(t), \quad (207b)$$

then we have

$$\left\langle F_{ij}(t) \right\rangle_R = 0, \quad (208)$$

$$\begin{aligned} \left\langle F_{ij}(t) F_{pq}(t') \right\rangle_R &= \sum_{t_c} \sum_{t'_c} \left\langle \left\{ \left[s_{ij}(t_c^+) - s_{ij}(t) \right] \delta(t - t_c^-) - A_{ij}(t) \right\} \right. \\ &\quad \left. \left\{ \left[s_{pq}(t'_c^+) - s_{pq}(t') \right] \delta(t' - t'_c^-) - A_{pq}(t') \right\} \right\rangle_R. \end{aligned} \quad (209)$$

We leave it to the reader to check that Eq. 209 can be transformed to

$$\begin{aligned}
\langle F_{ij}(t) F_{pq}(t') \rangle_R &= \delta(t-t') \sum_k \gamma_k \left\langle \left[s_{ij}(t^+, k) - s_{ij}(t) \right] \left[s_{pq}(t^+, k) - s_{pq}(t) \right] \right\rangle_R \\
&= \delta(t-t') (1-\delta_{ij})(1-\delta_{pq}) (2\gamma_i \delta_{iq} + 2\gamma_j \delta_{jp}) \langle s_{ij}(t) s_{pq}(t) \rangle_R \\
&= \delta(t-t') \langle s_{iq}(t) \rangle_R (1-\delta_{ij})(1-\delta_{pq}) (2\gamma_i \delta_{iq} + 2\gamma_j \delta_{jp}) \\
&= \delta(t-t') \langle s_{iq}(t) \rangle_R \left(\Gamma_{ij}^{\text{ph}} + \Gamma_{pq}^{\text{ph}} - \Gamma_{iq}^{\text{ph}} \right) \delta_{jp}. \tag{210}
\end{aligned}$$

6.3.3 Generalizations and Interpretations

We first want to know what happens if the transition-inducing reservoir and the phase shift-inducing reservoir are both present. We do not want to go through the calculations again, but it is straight-forward to check that the average equations just add, that the Langevin forces of the two reservoirs are uncorrelated, so that also the second-order moments of the Langevin forces just add. This property is not just a trivial consequence of the fact that the two types of reservoir forces are (obviously) independent. We introduce the following notation

$$R_{\text{in}}^{(i)} = \sum_k s_{kk} w_{ik} \quad ; \quad R_{\text{out}}^{(i)} = s_{ii} \sum_k w_{ki} = s_{ii} \Gamma_i \tag{211a}$$

$$\Gamma_i = \sum_k w_{ki} \quad ; \quad \Gamma_{ij} = \Gamma_{ji} = \frac{1}{2} (\Gamma_i + \Gamma_j) + \Gamma_{ij}^{\text{ph}}. \tag{211b}$$

Note that

$$\Gamma_{ii}^{\text{ph}} = 0 \quad ; \quad \Gamma_{ii} = \Gamma_i \quad ; \quad \Gamma_j = \frac{1}{2} [\Gamma_i + \Gamma_j + \Gamma_j + \Gamma_q - \Gamma_i - \Gamma_q]. \tag{211c}$$

We obtain from (197) and (206), (197) and (208), (200a) and (207a), (200b) and (210), respectively

$$\frac{ds_{ij}(t)}{dt} = A_{ij}(t) + F_{ij}(t) \tag{212a}$$

$$\langle F_{ij}(t) \rangle_R = 0 \tag{212b}$$

$$A_{ij}(t) = -s_{ij}(t) \Gamma_{ij} + \delta_{ij} R_{\text{in}}^{(i)}. \tag{212c}$$

$$\begin{aligned}
\langle F_{ij}(t) F_{pq}(t') \rangle_R &= \delta(t-t') \left[\delta_{jp} (\Gamma_{ij} + \Gamma_{pq} - \Gamma_{iq}) \langle s_{iq}(t) \rangle_R + \delta_{iq} \delta_{jp} \langle R_{\text{in}}^{(i)} \rangle_R \right. \\
&\quad \left. - \delta_{pq} \langle s_{ij}(t) \rangle_R w_{pj} - \delta_{ij} \langle s_{pq}(t) \rangle_R w_{ip} \right]. \tag{212d}
\end{aligned}$$

If we consider Eq. 212c for $i = j$, then we see that the constants w_{ij} should be interpreted as the transition probability from j into i . Indeed, first, Eq. 194 show that these transition probabilities are the (well-known) spontaneous and stimulated transition probabilities, and second, the average value of the operators $R_{in.}^{(i)}$ and $R_{out}^{(i)}$ are then, respectively, the average total atomic rate into level i and out of level i . The constant Γ_i is the transition probability out of level i ; Eq. 212c shows then that the average rate of change per unit time of a diagonal element s_{ii} is equal to the average of the total rate in minus the total rate out. The pure phase-shift reservoir induces no transitions and thus gives no contributions to $A_{ii}(t)$. If we consider (212c) for $i \neq j$, then we see that Γ_{ij} should be interpreted as the decay constant for the average of the off-diagonal element s_{ij} ; this decay constant is equal to one half of the total transition probability out of levels j and i plus the decay constant Γ_{ij}^{ph} caused by the pure phase-shift reservoir. The most important aspect of (212d) is that every second-order moment is expressed in terms of the decay constants and transition probabilities that enter into the averaged equations. This can be called a fluctuations-dissipation theorem. More specific interpretations are not always easy. Consider the various special cases. We set $\langle F_{ij}(t)F_{pq}(t') \rangle_R = \delta(t-t') \langle ij pq \rangle$. Note that $\langle ij pq \rangle^+ = \langle qpji \rangle$. In the following formulas one can derive all possible second-order moments by symmetry or by taking Hermitian conjugates. The i, j, p, q are considered to be different. Moments involving one level:

$$\langle iii i \rangle = \langle R_{in.}^{(i)} + R_{out}^{(i)} \rangle. \quad (213)$$

Moments involving two levels:

$$\langle i i j \rangle = \langle s_{ij} \rangle \Gamma_i ; \quad \langle i i j i \rangle = - \langle s_{ji} \rangle w_{ij} \quad (214a)$$

$$\langle i i j j \rangle = - \langle s_{ii} \rangle w_{ji} - \langle s_{jj} \rangle w_{ij} = - \langle R_{tr.}^{(ij)} + R_{tr.}^{(ji)} \rangle \quad (214b)$$

$$\langle i j i j \rangle = 0 \quad (214c)$$

$$\langle i j j i \rangle = \langle R_{in.}^{(i)} \rangle + \langle s_{ii} \rangle (\Gamma_j + 2\Gamma_{ij}^{ph}). \quad (214d)$$

Moments involving three levels:

$$\begin{aligned} \langle i i j p \rangle &= - \langle s_{jp} \rangle w_{ij} ; \quad \langle i j i p \rangle = 0 ; \quad \langle i j p i \rangle = 0; \\ \langle j i i p \rangle &= \langle s_{jp} \rangle (\Gamma_{ji} + \Gamma_{ip} - \Gamma_{jp}). \end{aligned} \quad (215)$$

Moments involving four levels:

$$\langle i j p q \rangle = 0. \quad (216)$$

The noise sources associated with diagonal elements (Eqs. 213 and 214b) can clearly be interpreted as shot-noise sources. Note also that these noise sources all commute.

Cross moments of noise sources associated with off-diagonal elements are more difficult to interpret. They are zero unless they are associated with transitions in and out of a common level. The most important one is given by Eq. 214d. If the material interacts only with the reservoir and reaches the steady state, then we have $\langle R_{in}^{(i)} \rangle = \langle R_{out}^{(i)} \rangle = \langle s_{ii} \rangle \Gamma_i$ and thus $\langle ijji \rangle = 2\Gamma_{ij} \langle s_{ii} \rangle$. If the level pair (ij) interacts also with a field system that transfers particles out of level i at an average rate $\langle R \rangle$, then, in the steady state, $\langle R_{in}^{(i)} \rangle = \langle R_{out}^{(i)} \rangle + \langle R \rangle$, so that $\langle ijji \rangle = 2\Gamma_{ij} \langle s_{ii} \rangle + \langle R \rangle$. The fact that these expressions have no simple classical analogue is related to the fact that s_{ij} has no classical analogue. Although we said that s_{ij} can be interpreted as being proportional to the positive or negative frequency components of the polarization, the fact that $s_{ij}s_{ji} = s_{ii}$ cannot be interpreted in classical terms. It is mainly this expression that is responsible for the strange forms of the noise-source moment $\langle ijji \rangle$. If we are willing to accept that relation, then it is clear that, e.g., the form $\langle ijji \rangle = 2\Gamma_{ij} \langle s_{ii} \rangle$ is a Nyquist type of formula. The cross moments of noise sources associated with diagonal and off-diagonal elements are even more difficult to interpret; note that they are all proportional to the average value of off-diagonal elements. If the material only interacts with its reservoir, then the steady-state value of these averages are zero; if, however, the material interacts also with a field system then these averages are not necessarily zero.

The formulas given here are less general than those derived by Lax¹⁰ because we have considered reservoirs that do not induce frequency shifts. In the transition reservoirs we could easily drop the assumption of symmetric distribution of the reservoir oscillators; instead of working with the Dirac delta function (Eq. 188), we should then have to work with the u-function (Eqs. 33); this generalization involves no difficulties. For the phase shift-inducing reservoir we adopted a phenomenological model and worked it out in an exact way. We could also have adopted a Hamiltonian that describes the interactions of a many-level system with a phonon reservoir and analyzed it in a similar way, as we did for the transition-inducing reservoir. Frequency shifts could then easily be included.

The relation between our derivation and those of other authors will be discussed in Section VIII.

6.4 EXACT EQUATIONS

We consider again the complete laser model, that is, the field mode, the material system, the loss system and the material reservoir, and their various interactions. The effect of the loss system on the field system has been described in Section III. It introduces a decay constant μ (Eq. 51) and a noise source $x_L^\pm(t)$ (Eq. 53); the properties of μ and $x_L^\pm(t)$ are clearly unaffected by the coupling of the field system to the material system. For the material system we cannot reach the same conclusion in such a simple way because the derivation of the Langevin forces for the material is not as straightforward as for the field. We have described the interaction of one many-level system

with its reservoir. Now we have a large number of many-level systems, and these are now coupled to the field. The derivation given in Section 6.3 is based on Appendix C. This Appendix is, however, much more general, and includes the more complicated case that we are now dealing with. We repeat the implications of Appendix C that we need here. Equation C.1 symbolizes in particular the following equations.

$$\begin{aligned} \frac{ds_o(t)}{dt} &= s_1(t)[x_1(t)+a_1] \quad ; \quad \frac{ds_1(t)}{dt} = s_2(t)[x_2(t)+a_2] \quad ; \\ \frac{ds'_o(t)}{dt} &= s'_1(t)[x'_1(t)+a'_1]. \end{aligned} \quad (217)$$

If the reservoir forces $x_2(t)$ and $x_1(t)$ are uncorrelated, then the only contribution to the average equation (A₁) comes from the drift term (a_1 -term) in the first-order term of the perturbation expansion, that is, it can immediately be identified from Eqs. 217. If the reservoir forces $x_1(t)$ and $x'_1(t)$ are uncorrelated, then the Langevin forces associated with s_o and s'_o are uncorrelated (this does not necessarily mean that they are independent).

If we consider the Hamiltonian describing the field system (Eq. 10), the material system (Eq. 15 summed over all particles), the interaction of field and material (Eq. 20), the loss system and its interaction with the field (Eqs. 28 and 30), an independent reservoir for each particle (e.g., Eq. 180 summed over all particles), then write the Heisenberg equations of motion, transform them to the form (217) by expressing the reservoir forces at t as a function of the reservoir forces a , $t = 0$ (see Eqs. 48 and 186), and apply the previous statement, then we reach the following conclusions.

(i) The average terms of the Langevin equations (A) are identical to those previously derived by considering each particle and the field system separately, except for additional terms derived from the interaction Hamiltonian between field and material (Eq. 20) through the Heisenberg formula.

(ii) The Langevin forces for each particle and for the field have the same second-order moments as previously derived (Eqs. 54, 55b, 212d). Also, the loss noise source and the noise sources for each particle are all uncorrelated with each other. We have thus far not been able to prove that they are also independent of each other. If this were so, we could immediately conclude that the noise sources driving the laser are Gaussian; indeed, as we shall see, the material noise sources enter as a linear combination of the noise sources of each particle. Since there are so many particles, the central limit theorem would ensure the Gaussian character of the material noise sources. We know that the loss noise source is Gaussian. If it is independent of the material noise sources, then they would all be joint-Gaussian.

We shall assume this independence property in the sequel. We expect it, if not to be exact, to be at least a very good approximation.

The equations (17), (35-39), and (211-212) contain enough information to obtain the

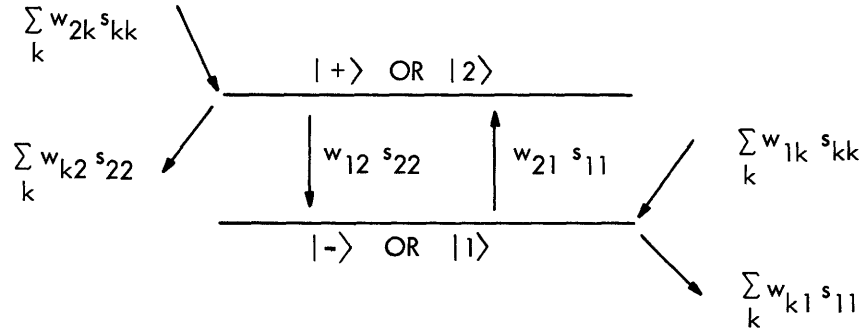


Fig. 10. Level pair (1, 2) of many-level system.

Langevin formulation of our laser model in the following form

$$\frac{da}{dt} + \mu a - \sum_j \kappa_j p_j^- = x_L^-; \quad \text{and Hermitian conjugate} \quad (218a)$$

$$\frac{dp_j^-}{dt} + \Gamma p_j^- - \kappa_j^* p_j a = x_j^-; \quad \text{and Hermitian conjugate} \quad (218b)$$

$$\frac{dp_j}{dt} + 2(\kappa_j a^+ p_j^- + \kappa_j^* p_j^+ a) + p_j \Gamma_{pp} - q_j \Gamma_{pq} = (r_p)_j + (x_p)_j \quad (218c)$$

$$\frac{dq_j}{dt} - p_j \Gamma_{qp} + q_j \Gamma_{qq} = (r_q)_j + (x_q)_j \quad (218d)$$

$$\left[\frac{ds_{kk}}{dt} + \Gamma_k s_{kk} - \sum_{k'} w_{kk'} s_{k'k'} + \frac{1}{2} (w_{k1} - w_{k2}) p - \frac{1}{2} (w_{k1} + w_{k2}) q = F_{kk} \right]_j \quad (218e)$$

$$\left[\sum_k s_{kk} + q \right]_j = 1. \quad (218f)$$

If $|2\rangle_j$ and $|1\rangle_j$ represent the upper and lower level of the level pair of particle j that interacts with the field mode, and if $|k\rangle_j$ represents any other level of that particle (Fig. 10), then the notations used in Eqs. 218 are

$$\begin{aligned} \Gamma &= \Gamma_{12} = \frac{1}{2} (\Gamma_1 + \Gamma_2) + \Gamma_{12}^{ph} ; \\ \Gamma_{pp} &= w_{21} + w_{12} + \Gamma_{qq} ; \quad \Gamma_{qq} = \frac{1}{2} \sum_k (w_{k1} + w_{k2}) \\ \Gamma_{pq} &= w_{21} - w_{12} + \Gamma_{qp} ; \quad \Gamma_{qp} = \frac{1}{2} \sum_k (w_{k1} - w_{k2}) \end{aligned} \quad (219a)$$

$$(r_p)_j = \sum_k (w_{2k} - w_{1k})(s_{kk})_j \quad ; \quad (r_q)_j = \sum_k (w_{2k} + w_{1k})(s_{kk})_j \quad (219b)$$

$$x_j^- = (F_{12})_j \quad ; \quad x_j^+ = (F_{21})_j \quad ; \quad (x_p)_j = (F_{22} - F_{11})_j \quad ; \quad (x_q)_j = (F_{22} + F_{11})_j$$

$$x_L^-, x_L^+ : \text{ See Eqs. 53 and 55 } \quad ; \quad F : \text{ See Eq. 212d} \quad (219c)$$

We shall also introduce

$$P = \sum_j p_j \quad ; \quad Q = \sum_j q_j \quad ; \quad S_{kk} = \sum_j (s_{kk})_j \quad ;$$

$$R_P = \sum_j (r_p)_j = \sum_k (w_{2k} - w_{1k}) S_{kk} \quad ; \quad R_Q = \sum_j (r_q)_j = \sum_k (w_{2k} + w_{1k}) S_{kk} \quad ; \quad (220a)$$

$$X_P = \sum_j (x_p)_j \quad ; \quad X_Q = \sum_j (x_q)_j \quad ; \quad X_{kk} = \sum_j (F_{kk})_j \quad ;$$

$$x_m^- = + \sum_j (\kappa_j x_j^-) (1/\Gamma) \quad ; \quad x^- = x_m^- + x_L^- \quad ; \quad (220b)$$

$$r_j = (\kappa_j a^+ p_j^- + \kappa_j^* p_j^+ a) \quad ; \quad R = \sum_j r_j \quad ; \quad R' = \sum_j |\kappa_j|^2 r_j \quad (220c)$$

$$\beta_L : \text{ See Eq. 31a } \quad ; \quad \beta_m = \langle P_- \rangle / \langle P \rangle$$

$$\epsilon = \Gamma / (\Gamma + \mu) \quad ; \quad 1 - \epsilon = \mu / (\Gamma + \mu). \quad (220d)$$

We see from Eq. 218c that r_j is the net rate for one particle at which electrons are transferred from the upper level (2) to the lower level (1) through radiation. The parameter R is the net rate for all particles.

If we operate with $\left(\frac{d}{dt} + \Gamma\right)$ on Eq. 218a and use Eq. 218b, we obtain

$$\frac{d^2 a}{dt^2} + (\Gamma + \mu) \frac{da}{dt} + \left(\Gamma \mu - \sum_j |\kappa_j|^2 p_j \right) a = \left(\frac{d}{dt} + \Gamma \right) x_L^- + \sum_j \kappa_j x_j^- \quad (221a)$$

or

$$C + \frac{da}{dt} + \epsilon \left(\mu - \Gamma^{-1} \sum_j |\kappa_j|^2 p_j \right) a = \epsilon x^- \quad (221b)$$

with

$$C = \frac{d}{dt} \left(\frac{da}{dt} - x_L^- \right) (\Gamma + \mu)^{-1} = \text{correction term.} \quad (221c)$$

From Eqs. 218c and 218a we deduce

$$\frac{dP}{dt} + 2 \frac{da^+ a}{dt} + 4 \mu a^+ a + P \Gamma_{pp} - Q \Gamma_{pq} = R_P + X_P + 2(a^+ x_L^- + x_L^+ a). \quad (222)$$

Finally, from (221b) and (222) we deduce

$$\frac{da^+a}{dt} + 2\epsilon \left(\mu - \Gamma^{-1} \sum_j |\kappa_j|^2 p_j \right) a^+a + a^+C + C^+a = \epsilon(a^+x^- + x^+a) \quad (223a)$$

$$\begin{aligned} \frac{dP}{dt} + 4 \left[\mu(1-\epsilon) + \epsilon \Gamma^{-1} \sum_j |\kappa_j|^2 p_j \right] a^+a + P\Gamma_{pp} - Q\Gamma_{pq} - 2(a^+C + C^+a) \\ = R_P + X_P + 2(1-\epsilon)(a^+x_L^- + x_L^+a) - 2\epsilon(a^+x_m^- + x_m^+a). \end{aligned} \quad (223b)$$

These equations can be complemented by

$$\begin{aligned} \frac{dQ}{dt} - P\Gamma_{qp} + Q\Gamma_{qq} &= R_Q + X_Q \\ \frac{dS_{kk}}{dt} + \Gamma_k S_{kk} - \sum_{k'} w_{kk'} S_{k'k'} + \frac{1}{2} (w_{k1} - w_{k2}) P - \frac{1}{2} (w_{k1} + w_{k2}) Q &= X_{kk} \\ \sum_k S_{kk} + Q &= N. \end{aligned} \quad (223c)$$

We mention also two exact steady-state relations. From (218c) and (222) we obtain

$$2 \langle R \rangle = \langle R_P - P\Gamma_{pp} + Q\Gamma_{pq} \rangle = 4\mu \langle a^+a \rangle - 4\mu\beta_L \quad (224a)$$

$$\langle R_Q + P\Gamma_{qp} - Q\Gamma_{qq} \rangle = 0. \quad (224b)$$

We now compile the important noise-source moments. We use the notation $\langle x(t)x'(t') \rangle = \delta(t-t') \langle xx' \rangle$. We obtain

$$\begin{aligned} \left\langle \left[x_L^+, x_L^- \right] \right\rangle &= -2\mu \quad ; \quad \left\langle \left[x_L^+, x_L^- \right]_+ \right\rangle = 2\mu(1+2\beta_L) \\ \left\langle \left[x_j^+, x_j^- \right] \right\rangle &= \left\langle \left[(2\Gamma - \Gamma_{pp})p + \Gamma_{pq}q + r_p \right]_j \right\rangle \stackrel{s \doteq s}{=} \langle 2\Gamma p_j + 2r_j \rangle \\ \left\langle \left[x_j^+, x_j^- \right]_+ \right\rangle &= \left\langle \left[2\Gamma q + \Gamma_{qp}p - \Gamma_{qq}q + r_q \right]_j \right\rangle \stackrel{s \doteq s}{=} \langle 2\Gamma q_j \rangle \\ \left\langle \left[x_m^+, x_m^- \right] \right\rangle &\stackrel{s \doteq s}{=} \left\langle \frac{2}{\Gamma} \sum |\kappa_j|^2 p_j + 2R'/\Gamma^2 \right\rangle \\ \left\langle \left[x_m^+, x_m^- \right]_+ \right\rangle &\stackrel{s \doteq s}{=} \left\langle \frac{2}{\Gamma} \sum |\kappa_j|^2 q_j \right\rangle \end{aligned}$$

$$\begin{aligned}
\langle X_Q X_Q \rangle &= \langle -P \Gamma_{qp} + Q \Gamma_{qq} + R_Q \rangle \stackrel{s \rightarrow s}{=} 2 \langle R_Q \rangle \\
\langle X_P X_P \rangle &= \langle P(-2\Gamma_{pq} + \Gamma_{qp}) + Q(2\Gamma_{pp} - \Gamma_{qq}) + R_Q \rangle \stackrel{s \rightarrow s}{=} \langle -2\Gamma_{pq} P + 2\Gamma_{pp} Q \rangle \\
\langle X_P X_Q \rangle &= \langle P \Gamma_{qq} - Q \Gamma_{qp} + R_P \rangle \\
\langle [x_m^-, X_P] \rangle &= +2\Gamma_{pp} \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma \quad ; \quad \langle [x_m^-, X_P]_+ \rangle = -2\Gamma_{pq} \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma \\
\langle [x_m^-, X_Q] \rangle &= -2\Gamma_{qp} \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma \quad ; \quad \langle [x_m^-, X_Q]_+ \rangle = 2\Gamma_{qq} \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma \\
\langle [x_m^-, X_{kk}] \rangle &= +(w_{k1} - w_{k2}) \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma \quad ; \\
\langle [x_m^-, X_{kk}]_+ \rangle &= -(w_{k1} + w_{k2}) \langle \Sigma \kappa_j p_j^- \rangle_R / \Gamma.
\end{aligned} \tag{225}$$

We know that the noise sources X_P , X_Q and X_{kk} can be interpreted as shot-noise sources. The rather unfamiliar forms given here result from the fact that $S_{22} = P_+$ and $S_{11} = P_-$ have been expressed in terms of $P = P_+ - P_-$ and $Q = P_+ + P_-$.

We have now formulated the exact equations that describe our laser model. These equations are complicated because (i) they are nonlinear, (ii) the population statistics can become very complicated, and (iii) the coupling constant $|\kappa_j|^2$ is in general a function of j ; this requires, in fact, that the equations for every particle should be retained. In most of the sequel we shall assume that $|\kappa_j|^2$ can be replaced by an appropriate average κ^2 .

6.5 SOLUTIONS IN SPECIAL CASES

We shall treat the steady state below threshold, the semiclassical analysis (that is, without noise sources) of the steady state above threshold, the phase fluctuations above threshold, and give a discussion of the general case and the intensity fluctuations in the case for which the material system consists of strictly two-level systems.

6.5.1 Operation Sufficiently below Threshold

If no noise sources were present (semiclassical theory), the steady-state solution would lead to a zero field, and the pump statistics would lead to a steady-state population inversion P_0 (c-number). From Eq. 221a it follows that the condition for operation below threshold is given by

$$\mu > \frac{\kappa^2}{\Gamma} P_0 = \gamma \quad ; \quad \beta_m = P_-^0 / P_0. \tag{226}$$

If the noise sources are present (quantum theory), we set

$$\sum_j |\kappa_j|^2 p_j = \kappa^2 (P_0 + P_1). \quad (227)$$

From Eq. 221a we obtain

$$\frac{d^2 a}{dt^2} + (\mu + \Gamma) \frac{da}{dt} + \Gamma(\mu - \gamma) a + \kappa^2 P_1 a = \frac{d}{dt} x_L^- + \Gamma x^- . \quad (228)$$

Let us now assume in a first approximation that (symbolically)

$$\kappa^2 P_1 / \Gamma \ll (\mu - \gamma) \quad (229)$$

We can then approximate (228) by

$$\frac{d^2 a}{dt^2} + (\mu + \Gamma) \frac{da}{dt} + \Gamma(\mu - \gamma) a = \frac{d}{dt} x_L^- + \Gamma x^- . \quad (230)$$

The properties of the noise sources x_L^\pm and $x^\pm = x_m^\pm + x_L^\pm$ were discussed in section 6.4. These noise sources are Gaussian, x_m^\pm is independent of x_L^\pm , and the second-order moments are proportional to Dirac delta functions. The proportionality constants, which are also the magnitude of the (frequency-independent) spectra of these noise sources, are given by (see Eqs. 225)

$$\langle x_L^+ x_L^- \rangle = 2\mu\beta_L \quad ; \quad [x_L^+, x_L^-] = -2\mu \quad (231a)$$

$$\langle x_m^+ x_m^- \rangle = \frac{2\kappa^2}{\Gamma} \langle P_+ + \frac{R}{2\Gamma} \rangle \quad ; \quad \langle [x_m^+, x_m^-] \rangle = \langle P + \frac{R}{\Gamma} \rangle \frac{2\kappa^2}{\Gamma} . \quad (231b)$$

We make now the following approximation in (231b)

$$\langle P_+ + \frac{R}{2\Gamma} \rangle = P_+^0 \quad ; \quad \langle P + \frac{R}{\Gamma} \rangle = P_0 . \quad (231c)$$

We shall show that this approximation is necessary in order to retain quantum-mechanical consistency with the approximation (229). It means that $\langle R/\Gamma \rangle$, that is, the average number of particles transferred through radiation from the upper to the lower level in the mean time Γ^{-1} , is very small compared with the average number of excited particles. We obtain then

$$\langle x_m^+ x_m^- \rangle = \frac{2\kappa^2}{\Gamma} P_+^0 = 2\gamma(1 + \beta_m) \quad ; \quad \langle [x_m^-, x_m^+] \rangle = -2\gamma \quad (231d)$$

$$\langle x^+ x^- \rangle = 2\mu\beta_L + 2\gamma(1 + \beta_m) \quad ; \quad \langle [x^-, x^+] \rangle = 2(\mu - \gamma) . \quad (231e)$$

From now on, we make no further approximations. We solve (230) with the technique

of Appendix A. If we set

$$\langle u(t+\tau)v(t) \rangle = \int_{-\infty}^{+\infty} \langle u(\omega)v(-\omega) \rangle e^{i\omega\tau} \frac{d\omega}{2\pi}, \quad (232)$$

then we obtain

$$a(\omega) = \frac{i\omega x_L^-(\omega) + \Gamma x^-(\omega)}{-\omega^2 + (\mu + \Gamma)i\omega + \Gamma(\mu - \gamma)} \quad (233)$$

and thus

$$\langle a^+(t+\tau)a(t) \rangle = \frac{1}{D} \left[\omega^2 \langle x_L^+ x_L^- \rangle + \Gamma^2 \langle x^+ x^- \rangle \right] \quad (234a)$$

$$\langle [a(t+\tau), a^+(t)] \rangle = \frac{1}{D} \left[\omega^2 \langle [x_L^-, x_L^+] \rangle + \Gamma^2 \langle [x^-, x^+] \rangle \right] \quad (234b)$$

$$\begin{aligned} D &= [\Gamma(\mu - \gamma) - \omega^2]^2 + (\mu + \Gamma)^2 \omega^2 \\ &= (\omega^2 + a^2)(\omega^2 + b^2). \end{aligned} \quad (234c)$$

The spectra of the noise sources are given in Eqs. 231a and 231e. The spectra of $\langle a^+(t+\tau)a(t) \rangle$ and $\langle [a(t+\tau), a^+(t)] \rangle$ are thus rather complicated functions. We shall not give here a general discussion of the shape of these spectra. We obtain correlation functions by using the following formula, which is valid for any A, B, and positive Re (a and b):

$$\int_{-\infty}^{+\infty} \frac{\omega^2 A + B}{(\omega^2 + a^2)(\omega^2 + b^2)} e^{i\omega\tau} \frac{d\omega}{2\pi} = \frac{e^{-a|\tau|}}{2a} \frac{a^2 A - B}{a^2 - b^2} + \frac{e^{-b|\tau|}}{2b} \frac{B - b^2 A}{a^2 - b^2} \quad (235a)$$

$$\int_{-\infty}^{+\infty} \frac{\omega^2 A + B}{(\omega^2 + a^2)(\omega^2 + b^2)} \frac{d\omega}{2\pi} = \frac{1}{2(a+b)} \left(A + \frac{B}{ab} \right). \quad (235b)$$

From Eq. 234c it follows that a and b are given by

$$a, b = \frac{1}{2}(\mu + \Gamma) \pm \frac{1}{2}[(\mu - \Gamma)^2 + 4\gamma\Gamma]^{1/2} = \frac{1}{2}(\mu + \Gamma) \pm \frac{1}{2}[(\mu + \Gamma)^2 - 4\Gamma(\mu - \gamma)]^{1/2}. \quad (236)$$

The appropriate values for A and B can be read from (234a) or (234b). With the help of these formulas, we can thus obtain the correlation functions $\langle a^+(t+\tau)a(t) \rangle$ and $\langle [a(t+\tau), a^+(t)] \rangle$. We do not want to discuss these results in detail. We want, however, to calculate $G_1(0)$ and $\langle [a(t), a^+(t)] \rangle$. Note that

$$a + b = \mu + \Gamma ; \quad ab = \Gamma(\mu - \gamma) ; \quad \epsilon = \frac{\Gamma}{\mu + \Gamma} \quad (237)$$

so that

$$\begin{aligned}
G_1(0) &= \langle a^+(t)a(t) \rangle = \frac{\langle x_L^+ x_L^- \rangle}{2(\Gamma + \mu)} + \frac{\Gamma^2 \langle x^+ x^- \rangle}{2\Gamma(\mu - \gamma)(\mu + \Gamma)} \\
&= (1 - \epsilon)\beta_L + \frac{\epsilon}{\mu - \gamma} [\mu\beta_L + \gamma(1 + \beta_m)]
\end{aligned} \tag{238}$$

$$\langle [a(t), a^+(t)] \rangle = (1 - \epsilon) + \frac{\epsilon}{\mu - \gamma} (\mu - \gamma) = 1. \tag{239}$$

We shall now discuss an approximation that is very good close to threshold, but not good enough to retain accuracy up to the quantum level. We assume that

$$\delta = \frac{\epsilon(\mu - \gamma)}{\mu + \Gamma} \ll 1. \tag{240}$$

We neglect $\frac{d^2 a}{dt^2}$ and $\frac{d}{dt} x_L^-$ in Eq. 181, and treat thus the equation

$$\frac{da}{dt} + \epsilon(\mu - \gamma)a = \epsilon x^-. \tag{241}$$

We obtain

$$\langle a^+(t+\tau)a(t) \rangle = \frac{\epsilon}{\mu - \gamma} [\mu\beta_L + \gamma(1 + \beta_m)] e^{-\epsilon(\mu - \gamma)|\tau|} \tag{242a}$$

$$\langle [a(t+\tau), a^+(t)] \rangle = \epsilon e^{-\epsilon(\mu - \gamma)|\tau|}. \tag{242b}$$

Equation 242 is clearly not correct. If we compare it with the exact results (234-239), we find that (242a) is correct to order δ , but that (242b) is only correct to order ϵ . This eventuates because the commutator does not contain the "enhancement" factor $\mu/(\mu - \gamma)$. The relatively large error in the commutator decays very fast with τ (that is, with decay constant $\mu + \Gamma$), so that the error is less important than (242b) for $\tau = 0$ would suggest (see Fig. 11).

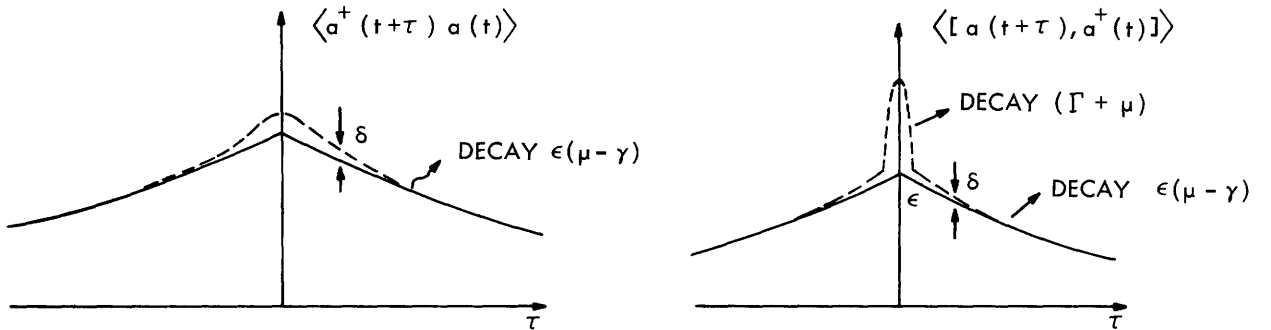


Fig. 11. Steady state below Threshold. --- Exact solution;
 ——— Approximation.

Since the noise sources are Gaussian, the steady-state field is also Gaussian and the higher order moments can be derived from the second-order moments given here. In particular,

$$G_2(\tau) = [G_1(\tau)]^2 + [G_1(0)]^2 \quad (243a)$$

$$G_2^1(\tau) = G_1(\tau) \left[G_1(\tau) + \left\langle [a(t+\tau), a^\dagger(t)] \right\rangle \right] + [G_1(0)]^2. \quad (243b)$$

From the approximate result (Eq. 242a) we find

$$\frac{G_2(\tau) - [G_1(0)]^2}{G_1(0)} = \epsilon \left[\frac{\mu}{\mu - \gamma} \beta_L + \frac{\gamma}{\mu - \gamma} (1 + \beta_m) \right] e^{-2\epsilon(\mu - \gamma)|\tau|}, \quad (243c)$$

where $\epsilon = \Gamma/(\Gamma + \mu)$. Thus we see that a finite material bandwidth ($\epsilon < 1$) has the effect of decreasing the bandwidth of the spectrum of the relative intensity fluctuations.

We also mention that the results for G_1, G_2, \dots, G_n can be rederived from an equivalent classical model consisting of an LC circuit loaded by a loss conductance and a frequency-dependent gain conductance. This can be understood from the fact that the "exact" linear equation (230) can be written in the Laplace formalism (with $\frac{d}{dt} = s$) in the following form

$$sa + \mu a - \frac{\gamma \Gamma}{s + \Gamma} a = x_L^- + \frac{\Gamma}{s + \Gamma} x_m^-. \quad (244)$$

The frequency-dependent factor $\gamma \Gamma/(s + \Gamma)$ describes the effect of finite material bandwidth on the gain conductance. For $\Gamma \rightarrow \infty$, it becomes equal to γ . We associate with μ and γ the Gaussian noise sources x_L^\pm and x_m^\pm with spectra given by

$$\langle x_L^+ x_L^- \rangle = \langle x_L^- x_L^+ \rangle = 2\mu\beta_L \quad ; \quad \langle x_m^+ x_m^- \rangle = \langle x_m^- x_m^+ \rangle = 2\gamma(1 + \beta_m). \quad (245)$$

These are the familiar Nyquist noise sources associated with thermal excitations and spontaneous emission. Because of the finite bandwidth of the material, these noise sources are coupled to the circuit as shown in (244). This is a well-known effect in classical noise theory. This circuit equation can again be solved either exactly or with the approximation discussed previously.

From the approximate equation (241a), one can deduce an equation for $a^\dagger a$. We obtain

$$\frac{da^\dagger a}{dt} + 2\epsilon(\mu - \gamma) a^\dagger a = \epsilon(a^\dagger x^- + x^+ a). \quad (246)$$

From (241a) and the techniques of Appendix C (see also section 4.8) we derive immediately

$$a^\dagger x^- + x^+ a = \epsilon[2\mu\beta_L + 2\gamma(1 + \beta_m)] + x(t) \quad ; \quad \langle x(t) \rangle = 0 \quad ;$$

$$\langle x(t+\tau)x(t) \rangle = \left\{ \langle a^\dagger a \rangle \left[4\mu\left(\frac{1}{2} + \beta_L\right) + 4\gamma\left(\frac{1}{2} + \beta_m\right) \right] + 2\mu\beta_L + 2\gamma(1 + \beta_m) \right\} \delta(\tau).$$

In section 4.8 we interpreted these expressions for the case $\epsilon = 1$, by means of a rate-equation model and shot noise. If however $\epsilon < 1$ (finite material bandwidth), then such an interpretation becomes difficult. Note that the effective net stimulated rate appears to be multiplied by the factor ϵ , the average thermal excitation and spontaneous emission drives by the factor ϵ^2 and the moment of the "shot-noise source" also by ϵ^2 . We have not tried to interpret these effects physically.

6.5.2 Semiclassical Analysis above Threshold

If we neglect all noise sources, we obtain for the steady state above threshold from Eqs. 218a and 218b

$$\mu a_s = \sum_j \kappa_j (p_j^-)_s ; \quad \Gamma \sum_j \kappa_j (p_j^-)_s - \sum_j |\kappa_j|^2 (p_j)_s a_s = 0, \quad (247)$$

where the subscript s stands for semiclassical. Thus for $a_s \neq 0$,

$$\frac{1}{\Gamma} \sum_j |\kappa_j|^2 (p_j)_s = \mu. \quad (248)$$

In order to obtain a_s , we have to solve Eqs. 218c-f. This is, in principle, a simple problem and thus we suppose that a_s has been determined. Note that $(p_j)_s$ is the population inversion as established by the field and the material reservoir. In some of the following formulas we shall assume that the presence of the noise sources does not change much to the average population inversion, that is, we shall set

$$\langle (p_j) \rangle = (p_j)_s. \quad (249)$$

The subscript or superscript s will then also stand for "saturated."

6.5.3 Phase Fluctuations above Threshold

One knows that "sufficiently" above threshold the field has small amplitude fluctuations around a large average value, and a slow phase drift. When the bandwidth of the spectra that we want to derive is very small compared with $\Gamma + \mu$, then we can neglect the term C in Eq. 221b. Above threshold the bandwidth of the phase fluctuations becomes rapidly very small (it is inversely proportional to power output). Therefore we may use

$$\frac{da}{dt} + \epsilon \left(\mu - \frac{1}{\Gamma} \sum_j |\kappa_j|^2 p_j \right) a = \epsilon x^-. \quad (250)$$

We set, with Lax,¹³ for sufficiently excited fields

$$a(t) = a_s e^{u(t)+iv(t)} ; \quad a^+(t) = a_s^+ e^{u(t)-iv(t)}, \quad (251)$$

where a_s is given by the semiclassical analysis (c-number), and where u and v are Hermitian operators. All moments of e^u are supposed to be small compared with the corresponding moments of a_s and a_s^+ . We obtain

$$\frac{du}{dt} + i \frac{dv}{dt} + \epsilon \left(\mu - \frac{1}{\Gamma} \sum |\kappa_j|^2 p_j \right) = \epsilon x^-(a)^{-1} \quad (252a)$$

$$\frac{du}{dt} - i \frac{dv}{dt} + \epsilon \left(\mu - \frac{1}{\Gamma} \sum |\kappa_j|^2 p_j \right) = \epsilon x^+(a^+)^{-1}. \quad (252b)$$

If we subtract these equations we obtain

$$\frac{dv}{dt} = \frac{\epsilon}{2i} [x^-(a)^{-1} - x^+(a^+)^{-1}] = n_v(t), \quad (253)$$

and thus

$$\langle [v(t+\tau) - v(t)]^2 \rangle = |\tau| \frac{\epsilon^2}{4} \langle x^- x^+ + x^+ x^- \rangle \langle a^+ a \rangle^{-1} = W |\tau| \quad (254a)$$

$$W = \frac{\epsilon^2}{\langle 4a^+ a \rangle} \left[\frac{2\kappa^2}{\Gamma} \bar{Q} + 2\mu(1+2\beta_L) \right] = \frac{\epsilon^2 \mu}{a_s^+ a_s} \left[\frac{1}{2} + \beta_m^s + \frac{1}{2} + \beta_L \right]. \quad (254b)$$

Why we may treat $(a^\pm)^{-1}$ as independent of x^\pm in the evaluation of the noise-source moment is well known from the theories of Appendix C. Spontaneous emission and thermal excitation have been neglected with respect to stimulated emission in the evaluation of these moments ($\langle n_v(t) \rangle = 0$, and $\langle a^+ a \rangle \approx \langle a a^+ \rangle$ in Eq. 254a). The phases present in $(a^\pm)^{-1}$ have no effect on the moments of the noise source n_v because in any moment $(a)^{-1}$ and $(a^+)^{-1}$ appear in products such that the phase drops out. We finally neglected amplitude fluctuations in $(a)^{-1}$ and $(a^+)^{-1}$ for the evaluation of the moments of n_v . Therefore, $(a)^{-1}$ and $(a^+)^{-1}$ become in effect constant c-numbers in n_v , so that n_v is Gaussian. Therefore,

$$\langle a_s^+ a_s e^{i[v(t+\tau) - v(t)]} \rangle = a_s^+ a_s e^{-\frac{1}{2} W |\tau|}. \quad (254c)$$

If we neglect the influence of the amplitude fluctuations on $G_1(\tau) = \langle a^+(t+\tau)a(t) \rangle$, then we have evaluated $G_1(\tau)$ in the expression (254c), but at the same time we have neglected "quantum corrections," that is, we find for $\langle a(t+\tau)a^+(t) \rangle$ the same result as for $G_1(\tau)$.

We see that $G_1(\tau)$ has a Lorentzian spectrum with full width at half-power equal to W . Since the power transmitted in the laser beam is given by

$$\bar{p} = 2\mu a_s^+ a_s \hbar \omega_0, \quad (255)$$

we obtain

$$W = \frac{\hbar\omega_0}{2\bar{p}} (2\mu\epsilon)^2 \left(\frac{1}{2} + \beta_m^s + \frac{1}{2} + \beta_L \right). \quad (256)$$

For $\epsilon \rightarrow 1$, this result agrees exactly with Eq. 174. Since $\epsilon = \Gamma/(\Gamma+\mu)$, we see that the effect of finite material bandwidth (Γ) is to decrease the laser bandwidth.

6.5.4 Discussion of the General Case

We shall assume that $|\kappa_j|^2$ can be replaced by an appropriate average κ^2 in $\sum |\kappa_j|^2 p_j$ and in $R' = \sum |\kappa_j|^2 r_j$, so that

$$\sum |\kappa_j|^2 p_j = \kappa^2 P \quad ; \quad R' = \kappa^2 R. \quad (257)$$

Equations 221a, 222 and 223c then form a complete set of nonlinear equations for the variables a , a^+ , a^+a , P , Q and S_{kk} . The moments of the noise sources can be derived from (225). The expression $a^+x_L^- + x_L^+a$ can be transformed by means of (218a) and the theory of Appendix C to

$$a^+x_L^- + x_L^+a = \langle x_L^+x_L^- \rangle + x_L(t), \quad (258)$$

and to evaluate the moments of $x_L(t)$ we treat a^\pm as independent of x_L^\pm . We set, with Haken,¹⁴

$$a(t) = e^{-i\theta(t)} [R_s + R_1(t)] \quad ; \quad a^+(t) = [R_s + R_1(t)] e^{i\theta(t)} \\ P = P_s + P_1(t) \quad ; \quad Q = Q_s + Q_1(t) \quad ; \quad S_{kk} = S_{kk}^{(s)} + S_{kk}^{(1)}(t), \quad (259)$$

where the parameters with subscript s are given by the semiclassical analysis. We linearize the equations in the parameters with subscript 1 and obtain

$$-2iR_s \left[\frac{d^2\theta}{dt^2} + (\mu + \Gamma) \frac{d\theta}{dt} \right] = e^{i\theta} \left[\frac{dx_L^-}{dt} + \Gamma x^- \right] - \left[\frac{dx_L^+}{dt} + \Gamma x^+ \right] e^{-i\theta} \quad (260a)$$

$$2 \left[\frac{d^2R_1}{dt^2} + (\mu + \Gamma) \frac{dR_1}{dt} - \kappa^2 R_s P_1 \right] = e^{i\theta} \left[\frac{dx_L^-}{dt} + \Gamma x^- \right] + \left[\frac{dx_L^+}{dt} + \Gamma x^+ \right] e^{-i\theta} \\ \frac{dP_1}{dt} + 4R_s \frac{dR_1}{dt} + 8\mu R_s R_1 + P_1 \Gamma_{pp} - Q_1 \Gamma_{pq} = R_P^{(1)} + X_P + 2(a^+x_L^- + x_L^+a) \\ \frac{dQ_1}{dt} - P_1 \Gamma_{qp} + Q_1 \Gamma_{qq} = R_Q^{(1)} + X_Q \\ \frac{dS_{kk}^{(1)}}{dt} + \Gamma_k S_{kk}^{(1)} - \sum_{k'} w_{kk'} S_{k'k'}^{(1)} + \frac{1}{2} (w_{k1} - w_{k2}) P_1 - \frac{1}{2} (w_{k1} + w_{k2}) Q_1 = X_{kk} \\ \sum_k S_{kk}^{(1)} + Q_1 = 0. \quad (260b)$$

The definitions of the Γ 's and R 's are given in Eqs. 219-220.

We note, first, that the phase factor $e^{\pm i\theta}$ in the right-hand side of Eqs. 260 has no influence on the moments of the noise sources (see Appendix C and section 6.5.3). Second, we may neglect the average value of the noise sources containing $e^{\pm i\theta}$ and a^{\pm} . By so doing we neglect the average drives of the thermal excitation and spontaneous emission. Below threshold these drives are responsible for the fact that the laser has any output at all; above threshold we can safely neglect these drives, as compared with the average stimulated emission drives, because R_s^2 is so large. Consistently, we neglect the difference between a^+a and aa^+ in evaluating the second-order moments of these noise sources. We thus treat $e^{\pm i\theta}$ and a^{\pm} effectively as c-numbers in the noise sources.

Note that all noise sources in (260b) commute among each other. For instance, from (225) and (247),

$$\begin{aligned} \left\langle \left[e^{i\theta} x_m^- + x_m^+ e^{-i\theta}, X_P \right] \right\rangle &= \frac{+2\Gamma_{pp}}{\Gamma} \left\langle \sum \kappa_j p_j^- e^{i\theta} - \sum \kappa_j^* p_j^+ e^{-i\theta} \right\rangle = \\ &= (+2\Gamma_{pp}/\Gamma)(\mu R_s - \mu R_s) = 0, \end{aligned} \quad (261)$$

so that if we solve Eqs. 260b by considering R_1, P_1, \dots either as operators or as c-numbers, we obtain the same results for the moments. The solution is, of course, in general very complicated, but since all noise-source moments can be calculated from (225), it is in principle straight forward if we apply the frequency-space techniques of Appendix A. Haken¹⁴ solved it for the case in which the material system consists of three-level systems. The spectrum of R_1 is then of fourth order in ω^2 . We shall consider in section 6.5.5 a special case in which the spectrum of R_1 is only of second order in ω^2 .

The equation for θ does not contain any of the other variables R_1, P_1, \dots . If we neglect the drift velocity of θ with respect to $(\mu + \Gamma)$, that is, if we neglect $(d^2\theta/dt^2)$ and (dx_L^\pm/dt) in Eq. 260a we obtain the solution discussed in section 6.5.3.

The noise source of (260a) is not independent, however, and does not commute with all of the noise sources of (260b). Indeed,

$$\begin{aligned} \left\langle \left[e^{-i\theta} x^- - x^+ e^{i\theta}, e^{-i\theta} x^- + x^+ e^{i\theta} \right] \right\rangle &= 2 \left\langle [x^-, x^+] \right\rangle = -4\mu \langle R \rangle / \Gamma \langle P \rangle \\ \left\langle \left[e^{-i\theta} x^- - x^+ e^{i\theta}, e^{-i\theta} x^- + x^+ e^{i\theta} \right]_+ \right\rangle &= 0 \end{aligned} \quad (262a)$$

$$\begin{aligned} \left\langle \left[e^{-i\theta} x^- - x^+ e^{i\theta}, X_P \right] \right\rangle &= (+2\Gamma_{pp}/\Gamma) 2\mu R_s = +2\Gamma_{pp} \langle R \rangle / \Gamma R_s \\ \left\langle \left[e^{-i\theta} x^- - x^+ e^{i\theta}, X_P \right]_+ \right\rangle &= 0, \end{aligned} \quad (262b)$$

where $\langle R \rangle = 2\mu R_s^2$ is the net rate at which particles are transferred from the upper level

to the lower level through radiation. It is not so difficult to show that all of these moments play approximately an equally important part in Eqs. 260. The ratio of the commutator in (262a) to the moment of the noise source driving θ or R_1 [which is $\langle [x^-, x^+]_+ \rangle = 4\mu \left(\frac{1}{2} + \beta_m^S + \frac{1}{2} \beta_L \right)$] is of order $\langle R \rangle / \Gamma \langle P \rangle$, that is, the ratio of the average number of particles transferred through radiation in the mean time Γ^{-1} to the number of inverted particles. Below threshold we neglected that ratio because its contribution to the ratio of commutator to anticommutator [which was of order $(\mu - \gamma) / \gamma$] was negligible. Above threshold, however, it is the only contribution; besides, it is possible to check that it is in fact of order $(\gamma - \mu) / \mu$ (in the notation of Section V). When it becomes of order 1 we are already operating very high above threshold.

If we neglect the influence of the amplitude fluctuations on G_1 , then we do not need the correlation between R_1 and θ , but we no longer have quantum accuracy (compare section 6.5.3). Of course, the influence of the amplitude fluctuations on G_1 becomes very small rapidly somewhat above threshold. The parameter θ does not enter into the expression for $G_2'(\tau)$ but does enter in the expression for $G_2(\tau)$. We can then calculate G_2' and neglect the difference between G_2 and G_2' . Close to threshold we can expect from the previous considerations (see also Section V) that this difference is small compared with the fluctuations in G_2' ; very high above threshold we can expect this difference to be of the same order as the fluctuations in G_2' , but these fluctuations are there very small, anyway.

We shall now discuss also the approximate equations that result from neglecting C^\pm in Eqs. 221 and 223. We have shown by our considerations for the steady state below threshold that such an approximation is justified if the bandwidth of the intensity fluctuations is very small compared with $(\mu + \Gamma)$. If we use (257), we have in (223) a complete set of equations for $a^\dagger a$, P , Q , and S_{kk} . The properties of the noise sources are given in Eqs. 225. The noise sources X_P , X_Q , and F_{kk} have been interpreted as shot-noise sources. The noise sources of the type $a^\dagger x^- + x^+ a$ are again transformed (compare section 6.5.1, Eqs. 246):

$$a^\dagger x_m^- + x_m^+ a = \epsilon \langle x_m^+ x_m^- \rangle + x_m(t) \quad ; \quad \langle x_m(t) \rangle = 0 \quad (263a)$$

$$\epsilon \langle x_m^+ x_m^- \rangle = \epsilon (2\kappa^2 / \Gamma) \langle P_+ + \frac{R}{2\Gamma} \rangle = \epsilon 2\mu \left(1 + \beta_m^S + \frac{\langle R \rangle}{2\Gamma \langle P \rangle} \right) \quad (263b)$$

$$\langle x_m(t+\tau) x_m(t) \rangle = \left[\langle a^\dagger a \rangle 4\mu \left(\frac{1}{2} + \beta_m^S \right) + 2\mu \left(1 + \beta_m^S + \frac{\langle R \rangle}{2\Gamma \langle P \rangle} \right) \right] \delta(\tau) \quad (263c)$$

$$a^\dagger x_L^- + x_L^+ a = \epsilon \langle x_L^+ x_L^- \rangle + x_L(t) \quad ; \quad \langle x_L(t) \rangle = 0 \quad (264a)$$

$$\epsilon \langle x_L^+ x_L^- \rangle = \epsilon 2\mu \beta_L \quad (264b)$$

$$\langle x_L(t+\tau) x_L(t) \rangle = \left[\langle a^\dagger a \rangle 4\mu \left(\frac{1}{2} + \beta_L \right) + 2\mu \beta_L \right] \delta(\tau) \quad (264c)$$

$$\begin{aligned}
\left\langle \left[x_m(t+\tau), X_{P,Q,S_{kk}}(t) \right]_{\pm} \right\rangle &= \left[\frac{\Gamma'}{\Gamma} \left\langle a^+ \sum \kappa_j p_j^- \pm \sum \kappa_j^* p_j^+ a \right\rangle \right] \delta(\tau) \\
&= \left[\frac{\Gamma'}{\Gamma} \left(\mu a_s^+ a_s \pm \mu a_s^+ a_s \right) \right] \delta(\tau)
\end{aligned} \tag{265}$$

where $\langle R \rangle$ is the net rate at which particles are transferred from upper to lower state through radiation, and Γ' symbolizes the appropriate Γ 's as given in Eqs. 225.

The average drives in (263b) and (264b) can be interpreted for $\epsilon = 1$ as spontaneous emission and thermal excitation. Note that the spontaneous emission is proportional to $\langle P_+ + \frac{R}{2\Gamma} \rangle$ and not to $\langle P_+ \rangle$. For $\epsilon < 1$, these drives are multiplied by a factor ϵ , and the coupling to $a^+ a$ and P as described in (223) gives rise to another factor ϵ or $\epsilon - 1$. We have not tried to interpret this physically. These average drives were extremely important below threshold because they were responsible for the fact that the laser had any output at all; above threshold we may safely neglect them, as compared with the stimulated average drives (proportional to $a_s^+ a_s$).

Equations 263c can be interpreted as moments of shot-noise sources (see section 4.8 for an interpretation of the independent rates). They are now coupled to $a^+ a$ and P , as shown in Eqs. 223. If $\epsilon < 1$, this coupling introduces again factors ϵ^2 , $(1-\epsilon)^2$ or $\epsilon(1-\epsilon)$. We have not tried to interpret that effect physically. The factor 2 in (223b) simply shows that the "shot" for P is 2. The correction terms in (263c) and (264c) will again be neglected above threshold.

Equation 265 shows that the shot-noise sources associated with radiative transfer and pump or nonradiative transfer commute but are not uncorrelated. It shows also that all noise sources in (223) commute, so that the solution is not affected by the fact that $a^+ a$, P , Q , and S_{kk} are operators.

In order to solve Eqs. 223, we set

$$a^+ a = a_s^+ a_s + n_1(t) \quad ; \quad P = P_s + P_1(t) \quad ; \quad \dots, \tag{266}$$

where $a_s^+ a_s$, P_s , ... are given by the semiclassical analysis. We linearize the equations in n_1 , P_1 , ..., and apply the techniques of Appendix A in order to obtain the second-order moments, in particular, the second order moment of $n_1(t)$. We have then calculated

$$G_2^1(\tau) = \left(a_s^+ a_s \right)^2 + \langle n_1(t+\tau) n_1(t) \rangle. \tag{267}$$

Again, the calculations can become rather involved unless one simplifies the model for the material statistics.

6.5.5 Strictly Two-level System and Highly Inverted System

We shall restrict ourselves to the case in which the rate-equation formulation

discussed in section 6.5.4 is valid, that is, we assume that the bandwidth of the intensity fluctuations is small compared with $(\mu + \Gamma)$. This condition should be checked a posteriori.

We first discuss the case in which the material system consists of strictly two-level systems. From Eqs. 219, 220, and 224 it follows that

$$Q = N \quad ; \quad \Gamma_{pp} = w_{21} + w_{12} \quad ; \quad \Gamma_{pq} = w_{21} - w_{12} \quad ; \quad R_P = 0 \quad ; \quad R_Q = 0 \quad ; \quad (267a)$$

$$2 \langle R \rangle = 4\mu \langle a^\dagger a \rangle - 4\mu\beta_L = Q\Gamma_{pq} - \langle P \rangle \Gamma_{pp}. \quad (267b)$$

We set

$$\Gamma_{pq}Q = \Gamma_{pp}P_0, \quad (267c)$$

so that P_0 is the population inversion established by the material reservoir alone. Equations 223 now read

$$\frac{da^\dagger a}{dt} + 2\epsilon[\mu - (\kappa^2/\Gamma)P] a^\dagger a = \epsilon(a^\dagger x^- + x^+ a) = x_1 \quad (268a)$$

$$\begin{aligned} \frac{dP}{dt} + 4[\mu(1-\epsilon) + \epsilon(\kappa^2/\Gamma)P] a^\dagger a + \Gamma_{pp}(P - P_0) \\ = X_P + 2(1-\epsilon)(a^\dagger x_L^- + x_L^+ a) - 2\epsilon(a^\dagger x_m^- + x_m^+ a) = X_P + x_2 \end{aligned} \quad (268b)$$

$$\epsilon = \Gamma/(\Gamma + \mu). \quad (268c)$$

We neglect the average value of the noise sources x_1 and x_2 (and thus the $\mu\beta_L$ term in Eq. 267b), and the difference between $\langle a^\dagger a \rangle$ and $\langle aa^\dagger \rangle$ in their second-order moments. We set

$$P = P_s + P_1 \quad ; \quad a^\dagger a = R_s^2 + n_1 \quad (269a)$$

$$\mu = (\kappa^2/\Gamma)P_s \quad ; \quad \gamma = (\kappa^2/\Gamma)P_0 \quad ; \quad 1 + 2\beta_m^s = Q/P_s \quad ; \quad (269b)$$

$$4\mu R_s^2 = \Gamma_{pp}(P_0 - P_s) = \frac{\gamma - \mu}{\kappa^2/\Gamma \Gamma_{pp}} \quad (269c)$$

Note that the term $(\kappa^2/\Gamma \Gamma_{pp})$ plays the part of α in Section V. We linearize Eqs. 268 in P_1 and n_1 and obtain

$$\frac{dn_1}{dt} - a_{np}P_1 = x_1 \quad ; \quad \left(a_{np} = \epsilon \frac{\gamma - \mu}{2\mu} \Gamma_{pp} \right) \quad ; \quad (270a)$$

$$\frac{dP_1}{dt} + 4\mu n_1 + a_{pp}P_1 = X_P + x_2 \quad ; \quad \left[a_{pp} = \Gamma_{pp} \left(1 + \epsilon \frac{\gamma - \mu}{\mu} \right) \right]. \quad (270b)$$

We have for the noise-source moments

$$\langle x_1 x_1 \rangle = 4\mu R_s^2 \epsilon^2 (1 + \beta_m^s + \beta_L) \quad (271a)$$

$$\langle x_2 x_2 \rangle = 4\mu R_s^2 \left[4(1-\epsilon)^2 \left(\frac{1}{2} + \beta_L \right) + 4\epsilon^2 \left(\frac{1}{2} + \beta_m^s \right) \right] \quad (271b)$$

$$\langle x_1 x_2 \rangle = 4\mu R_s^2 \left[2\epsilon(1-\epsilon) \left(\frac{1}{2} + \beta_L \right) - 2\epsilon^2 \left(\frac{1}{2} + \beta_m^s \right) \right] \quad (271c)$$

$$\begin{aligned} \langle X_P X_P \rangle &= 2\Gamma_{pp} Q - 2\Gamma_{pq} P_s = 4(w_{12} P_+^s + w_{21} P_-^s) \\ &= 4\mu R_s^2 \frac{4\mu}{\gamma - \mu} \left(\frac{1}{2} + \beta_m^s - \frac{1}{2} \frac{P_o}{Q} \right). \end{aligned} \quad (271d)$$

From Eqs. 265 we obtain

$$\begin{aligned} \langle [x_1, X_P]_+ \rangle &= -\frac{1}{2} \langle [x_2, X_P]_+ \rangle = -\epsilon 4\mu R_s^2 (\Gamma_{pq}/\Gamma) \\ &= -\epsilon 4\mu R_s^2 (\Gamma_{pp}/\Gamma) (P_o/Q) \end{aligned} \quad (271e)$$

$$\langle [x_1, X_P]_- \rangle = \langle [x_2, X_P]_- \rangle = 0. \quad (271f)$$

The shot-noise source (271d) was obtained by means of a phenomenological collision model in the thesis on which this report is based. The noise sources (271a-c), for $\epsilon = 1$, are shot-noise sources associated with photon and population-inversion transfers between the field system, the loss system, and the material system; for $\epsilon < 1$ they are more difficult to interpret.

If we apply the techniques of Appendix A to Eqs. 270, we find

$$(1/R_s^2) \langle n_1(\omega) n_1(-\omega) \rangle = \frac{A\omega^2 + B}{D(\omega)}, \quad (272)$$

where

$$A = (1/R_s^2) \langle x_1 x_1 \rangle = \epsilon^2 4\mu (1 + \beta_m^s + \beta_L) \quad (273a)$$

$$\begin{aligned} B &= (1/R_s^2) \left[a_{pp}^2 \langle x_1 x_1 \rangle + a_{np}^2 \langle (X_P + x_2)^2 \rangle + 2a_{pp} a_{np} \langle x_1 (x_2 + X_P) \rangle \right] \\ &= \epsilon^2 \Gamma_{pp}^2 4\mu \left[\frac{1}{2} + \beta_m^s + \left(\frac{1}{2} + \beta_L \right) (\gamma/\mu)^2 + \frac{\gamma - \mu}{\mu} \left(\frac{1}{2} + \beta_m^s - \frac{P_o}{2Q} - \frac{P_o}{2Q} \frac{\Gamma_{pp}}{\Gamma} \right) \right] \end{aligned} \quad (273b)$$

$$\begin{aligned} D(\omega) &= [-\omega^2 + 2\epsilon(\gamma - \mu) \Gamma_{pp}]^2 + \omega^2 \Gamma_{pp}^2 \left(1 + \epsilon \frac{\gamma - \mu}{\mu} \right)^2 \\ &= (\omega^2 + a^2)(\omega^2 + b^2) \end{aligned} \quad (274a)$$

$$a, b = \frac{1}{2} \Gamma_{pp} \left(1 + \epsilon \frac{\gamma - \mu}{\mu} \right) \pm \frac{1}{2} \left[\left(1 + \epsilon \frac{\gamma - \mu}{\mu} \right)^2 - 8\epsilon \frac{\gamma - \mu}{\Gamma_{pp}} \right]^{1/2} \quad (274b)$$

$$ab = 2\epsilon \Gamma_{pp} (\gamma - \mu) \quad ; \quad a + b = \Gamma_{pp} \left(1 + \epsilon \frac{\gamma - \mu}{\mu} \right). \quad (274c)$$

Since $\text{Re}(a)$ and $\text{Re}(b)$ are positive, we can use the formulas (232) and (235) to deduce the correlation function $\langle n_1(t+\tau) n_1(t) \rangle$. Note that a and b can eventually become complex, so that then the correlation function contains oscillatory factors.

We discuss also the case of a highly inverted material system. One neglects P_- with respect to P_+ . If one assumes that the population of the levels that supply P_+ does not fluctuate, then we obtain

$$\frac{da^+ a}{dt} + 2\epsilon \left[\mu - (\kappa^2 / \Gamma) P_+ \right] a^+ a = x_1 \quad (275a)$$

$$\frac{dP_+}{dt} + 2 \left[\mu(1-\epsilon) + \epsilon(\kappa^2 / \Gamma) P_+ \right] a^+ a + \Gamma_+ (P_+ - P_+^0) = X_+ + \frac{1}{2} x_2 \quad (275b)$$

with

$$\langle X_+ X_+ \rangle = \Gamma_+ (P_+^0 + P_+^s) \quad ; \quad \langle [x_1, X_+]_+ \rangle = -\frac{1}{2} \langle [x_2, X_+]_+ \rangle = \epsilon 4\mu R_s^2 (\Gamma_+ / \Gamma) \quad (275c)$$

If one goes through the same type of analysis, one finds the result (272) in which A and D are respectively obtained from (273a) and (274) by setting $\beta_m^s \rightarrow 0$ and $\Gamma_{pp} \rightarrow \Gamma_+$. The parameter B is now given by

$$B = \epsilon^2 \Gamma_+^2 4\mu \left[(1 + \beta_L) (\gamma / \mu)^2 + \frac{\gamma - \mu}{\mu} \frac{\Gamma_+}{\Gamma} \right]. \quad (276)$$

This result was obtained by McCumber⁹ for the case $\epsilon = 1$ and $\beta_L = 0$, by assuming the equivalence of the rate equation and shot-noise model.

In general, the spectrum (272) can have maxima for $\omega \neq 0$. McCumber discusses such cases in great detail. One should, of course, check that ω near that maximum is much smaller than $(\Gamma + \mu)$, otherwise this modified rate-equation model is no longer valid and one has to go back to the general equations (260b). They would give rise to a spectrum that is one order in ω^2 higher. If

$$\Gamma_{pp} \gg 2\epsilon(\gamma - \mu) \left[1 + \epsilon \frac{\gamma - \mu}{\mu} \right]^{-2}, \quad (277)$$

then the spectrum (272) reduces to a Lorentzian with full bandwidth at half-power given by

$$4\epsilon(\gamma - \mu) \left[1 + \epsilon \frac{\gamma - \mu}{\mu} \right]^{-1} \quad (278)$$

and magnitude at zero frequency given by $B/D(0)$. This magnitude is thus independent

of ϵ . The bandwidth is approximately proportional to ϵ .

6.6 CONCLUSIONS

We have considered a laser model in which there are no restrictions on the relative magnitude of the "cold-cavity" bandwidth (2μ), the material bandwidth (Γ), and the decay constants of the populations. We have obtained the operator noise-source formulation for this laser model. We have shown that in the steady state "sufficiently" below threshold solutions can be obtained with complete quantum accuracy. The solution "sufficiently" above threshold is essentially a straightforward problem if one is willing to forego complete quantum accuracy. The solution for the phase fluctuations is not involved because it is independent of the population statistics. The solution for the intensity fluctuations can become very involved. If its bandwidth is small compared with $(\Gamma + \mu)$, they can be obtained from a "modified" rate equation and shot-noise model (Eqs. 223); otherwise one has to work with the general equations (260b).

VII. FIELDS IN THE LASER BEAM

Since measurements of the laser fields are not performed on the fields inside the laser cavity (a-field), but on the fields in the beam outside the laser cavity (E-field), it is important to calculate the moments of the fields in the laser beam. We shall summarize here the results of Section VI of the thesis on which this report is based.

It is possible to describe the outside-space fields in terms of the modes of an artificially closed "outside-space cavity." These modes act on the field mode of the laser cavity in the same way as the harmonic oscillators of the loss reservoir discussed in Sections III-VI of this report. As soon as the solution for the field inside the cavity is obtained, it is possible to solve for the amplitudes of the outside-space modes. We obtained terms proportional to the a-field (amplitude of the inside-field mode) and terms proportional to the loss noise source described in Sections III-VI. From these expressions one can then calculate the Glauber functions of the fields in the laser beam, provided one is able to obtain expressions for the cross correlation functions between the loss noise sources and the a-fields. We have shown for the laser model of Section V, that if the temperature of the outside space is zero, the Glauber functions $G_1(E, \tau)$ and $G_2(E, \tau)$ of the fields in the laser beam are proportional to $G_1(a, \tau)$ and $G_2(a, \tau)$ of the field inside the laser cavity.

We proved a free-field theorem: "If we artificially close a source-free part of space with an artificial wall (with space-coordinates r_s); if we consider space points r sufficiently far from the wall, and time points t both in the past and the future of a reference time 0, such that $|t| < |r - r_s|/c$ for all r_s ; then the e.m. fields in these space-time points can be expanded in terms of the solenoidal modes of this artificial cavity; the amplitudes are boson creation and annihilation operators that have the free-field time dependence starting from the reference time 0. The whole range of t from past to present is the time that a perturbation needs to travel from the observation point (r) to the wall (r_s) and back." If we thus specify the density matrix at $t = 0$ in the Hilbert space of these amplitudes (e.g., by a Glauber $\rho(\{a\})$ representation), then the field at r is completely determined over the above-mentioned time range. Such a representation is, however, not necessarily the most convenient one ($\rho(\{a\})$ is a multidimensional function). We used, for instance, a representation in terms of the a-field and the loss noise source. We showed for the laser of Section V that the fields in the laser beam obeyed an implication of the free-field theorem, that is, the positive (negative) frequency components of the field in the laser beam commute over the above-mentioned time range.

VIII. CONCLUSIONS

8.1 SUMMARY OF THE MAIN RESULTS

We have developed the "operator noise-source technique" to solve the problem of quantum noise in the laser oscillator. This quantum technique is formally analogous to classical Langevin techniques. We have defined the characteristic Gaussian property of moments of operators, and we have used this concept to extend the operator noise-source technique to higher order moments.

In our laser models the field system is coupled to the material system; the field system is also coupled to a field reservoir – the loss system; the material system is also coupled to a material reservoir. The operator noise-source technique involves essentially the elimination of the unknown variables of the reservoirs. We have treated two different laser models. In the first model the material bandwidth and decay constants of the populations are assumed to be much larger than the cold-cavity bandwidth; this has allowed us to treat the material as a reservoir for the field. In the second model no restrictions are put on the relative magnitude of these various decay constants.

In the first model we have been able to obtain solutions with complete quantum accuracy. We have shown that in the steady state below threshold, the moments of normally ordered products of the creation and annihilation operators of the field mode can be derived from an equivalent classical field model that contains the familiar Gaussian Nyquist noise sources associated with linear loss and spontaneous emission. The moments of the photon-number operator can be derived from an equivalent classical-particle model by means of rate equations and shot noise. The rates were identified as spontaneous and stimulated emission and absorption into and out of the material system and loss system. The analysis of this first model above threshold led to a proof of the equivalence of a classical model, assumed by Haus,³ in which classical equations contain noise sources that correctly predict the field moments below threshold. We obtained small "saturation" and "quantum" corrections. These are too small to have been detected experimentally.

In the second model we have been able to derive results for the steady state below threshold, with complete quantum accuracy. We have shown that the moments of normally ordered products can be derived from a classical field model that contains the familiar Gaussian Nyquist sources associated with a linear loss conductance, and a linear, but frequency-dependent, gain conductance. We have shown that the moments of the photon-number operator can be derived from a "modified" rate equation and shot-noise model, provided the bandwidths of these moments are small compared with the sum of the cold-cavity bandwidth and the material bandwidth. This model does not lead to complete quantum accuracy if the time difference in the moments is smaller than the inverse of that sum. Above threshold, we have first discussed the small influence of the

commutators of some of the noise sources; we have then neglected these commutators, and thus neglected "quantum" corrections, that is, no solution for the field commutator can be obtained. We have obtained the phase fluctuations: for large material bandwidth, they correspond exactly to results obtained for the first laser model. To obtain the solution for the intensity fluctuations, one has to solve a set of coupled linearized equations of at least third-order (Eqs. 260b). If the bandwidth of the intensity fluctuations is smaller than the sum of the bandwidths of the cold cavity and the material, then one can obtain solutions by means of a "modified" rate equation and shot-noise model (Eqs. 223) of at least second-order. We solved that model for two second-order cases. If the material bandwidth is much larger than the cold-cavity bandwidth, then this "modified" rate-equation model reduces to the rate-equation model used by McCumber.⁹

8.2 DISCUSSION OF THE RESULTS BY OTHER AUTHORS

We shall first discuss the noise-source formulations, then the results obtained with these formulations, and finally the results obtained with other quantum techniques.

The noise-source formulation for our first laser model was derived by us completely independently. The form of the noise-source formulation of the interaction between a many-level system and its reservoirs, which we needed for our second laser model, was suggested to us by Lax,¹⁰ and by Haken and Weidlich.¹¹ These authors essentially assumed that the Langevin forces introduced by the reservoir have Dirac delta-function correlations. They did not eliminate the unknown reservoir parameters, but assumed that the reservoir parameters were known. They called these assumptions the Markovian assumption. We have tried to show that it is possible to adopt a reservoir model that leads without approximations and in a completely consistent way to the results obtained by these authors. Sauermann¹² has also given an exact derivation. His method is based on a disentanglement theorem, but it is applicable only if the harmonic oscillators of the reservoir are in the vacuum state. In this case it is possible to order the "x" and "s" in Eq. C. 1 (Appendix C) in such a way that the only contribution to the average equation comes from the "a-term" in Eq. C. 1. That is the reason for the success of his method.

All of the results for our first laser model have been derived independently. We have retained complete quantum-mechanical consistency in that model, that is, we were able to derive an expression for the field commutator. In our second model we have not retained such quantum accuracy. We shall now discuss the results for the second model. Lax¹³ has derived results for the field moments below threshold, and for the phase fluctuations above threshold. He includes detuning effects. Otherwise our results agree with his. Our contribution to the study of the steady state below threshold is the derivation and discussion of the "modified" rate-equation approximation. We have also given an equivalent circuit for the exact equations. For the phase fluctuations above threshold we have discussed qualitatively the influence of the noise-source commutators. Haken¹⁴ obtained results for the intensity fluctuations above threshold for the case of a fourth-order system, that is, a system described by a second-order differential equation for

the field and by two variables for the material system (Eqs. 260b). We have discussed qualitatively the influence of the noise-source commutators on the Glauber functions and the photon-number correlation function. We have only mentioned results for second-order systems, that is, systems in which the field is described by a "modified" rate equation (Eqs. 223), and the material by one variable. We have concentrated on such cases because we wanted to verify the results of McCumber.

McCumber⁹ essentially assumed the "equivalence" of the rate equation and shot-noise model. With the operator noise source formulation we have been able to prove the "equivalence" of his model if the material bandwidth is much larger than the cold-cavity bandwidth and the bandwidth of the intensity fluctuations. If the cold-cavity bandwidth is of the same order or larger than the material bandwidth, then the rate-equation model has to be "modified." If the intensity-fluctuations bandwidth is not much smaller than the sum of cold-cavity and material bandwidths, then the rate equations, modified or not, are no longer valid.

Korenmann¹⁶ extended the Green's function technique to the case of steady states that are not the thermal equilibrium state, and applied this new technique to the laser oscillator above threshold. It is clear that the standard Green's function technique is a moment technique and not a noise-source technique. In a moment technique the equations of motion of moments involving two times are essentially the same as for moments at a single time, and different solutions result from different initial or boundary conditions. We did not study Korenmann's technique sufficiently to be able to discuss its correspondence with the noise-source techniques.

Scully and his co-workers derived a solution for the equations of motion of the density matrix. Since the phase fluctuations describe a moment involving two different times, it is clear that Scully needed a moment technique to obtain his result for the phase fluctuations.

We finally discuss our theory on the Gaussian character of moments of operators. The only work that we have found which is related to this problem is by Glauber.⁵ He defines the Gaussian character of normally ordered moments of free fields by means of a $\rho(\alpha)$ representation. Not that a $\rho(\alpha)$ representation is not sufficient to describe non-free fields, that is, fields in the source. Our derivation is not limited to boson creation and annihilation operators, to normally ordered products, and to free fields.

APPENDIX A

Equations of Motion for Second-order Moments

Throughout this report we have given results for second-order moments. We shall show the method by which these results were obtained. Consider the following equations of motion for the operators u and v :

$$\frac{du(t+\tau)}{dt} + au(t+\tau) = n_u(t+\tau) \quad (\text{A. 1})$$

$$\frac{dv(t)}{dt} + bv(t) = n_v(t), \quad (\text{A. 2})$$

where $n_u(t)$ and $n_v(t)$ are noise-source operators with known moments. We cross-multiply these equations and take expectation values

$$\left\langle \frac{du(t+\tau)}{dt} \frac{dv(t)}{dt} \right\rangle + a \left\langle u(t+\tau) \frac{dv(t)}{dt} \right\rangle + b \left\langle \frac{du(t+\tau)}{dt} v(t) \right\rangle + ab \langle u(t+\tau)v(t) \rangle = \langle n_u(t+\tau)n_v(t) \rangle. \quad (\text{A. 3})$$

We assume now that the noise sources are stationary and that the system is stable ($a, b > 0$), so that the output reaches a steady state. We use the following notation in the steady state:

$$\phi_{uv}(\tau) = \langle u(t+\tau)v(t) \rangle \quad ; \quad \phi_{nuv}(\tau) = \langle n_u(t+\tau)n_v(t) \rangle. \quad (\text{A. 4})$$

These functions are independent of t . Therefore

$$\dot{\phi}_{uv}(\tau) = \left\langle \frac{du(t+\tau)}{d\tau} v(t) \right\rangle = \left\langle \frac{du(t+\tau)}{dt} v(t) \right\rangle = - \left\langle u(t+\tau) \frac{dv(t)}{dt} \right\rangle \quad (\text{A. 5})$$

$$\ddot{\phi}_{uv}(\tau) = - \left\langle \frac{du(t+\tau)}{d\tau} \frac{dv(t)}{dt} \right\rangle = - \left\langle \frac{du(t+\tau)}{dt} \frac{dv(t)}{dt} \right\rangle \quad (\text{A. 6})$$

If we use (A. 5) and (A. 6) in (A. 3), we get

$$-\ddot{\phi}_{uv}(\tau) + (b-a) \dot{\phi}_{uv}(\tau) + ab\phi_{uv}(\tau) = \phi_{nuv}(\tau). \quad (\text{A. 7})$$

For reasons of simplicity, we now suppose that $\phi_{nuv}(\tau) = A\delta(\tau)$. The roots of the characteristic equation of (A. 7) are $-a$ and b . For $\tau > 0$, we choose the decaying solution $B_1 \exp(-a\tau)$; for $\tau < 0$, we choose the solution $B_2 \exp(b\tau)$. Matching at the origin leads to $B_1 = B_2 = A/(a+b)$; therefore,

$$\begin{aligned} \phi_{uv}(\tau) &= [A/(a+b)] \exp(-a\tau), \quad (\tau > 0) \\ \phi_{uv}(\tau) &= [A/(a+b)] \exp(b\tau), \quad (\tau < 0). \end{aligned} \quad (\text{A. 8})$$

Under conditions (i) steady state, and (ii) correlation functions remaining finite for

$|\tau| \rightarrow \infty$, we found a unique solution from (A.1) and (A.2). These solutions still hold in the limiting case when either a or b goes to $+0$. If a and b are both zero, it is easier to integrate first (A.1), (A.2) and then take cross products.

This technique can also be formulated in frequency space. If we define

$$\sqrt{2T} u(\omega) = \int_{-T}^T u(t) \exp(-i\omega t) dt \quad ; \quad (\text{for } T \rightarrow \infty) \quad (\text{A. 9})$$

$$\phi_{uv}(\omega) = \int_{-T}^T \phi_{uv}(\tau) \exp(-i\omega\tau) d\tau \quad ; \quad (\text{for } T \rightarrow \infty), \quad (\text{A. 10})$$

then (for $T \rightarrow \infty$)

$$\langle u(\omega)v(-\omega) \rangle = (1/2T) \int_{-T}^T dt' \int_{-T}^T dt \langle u(t)v(t') \rangle \exp[-i\omega(t-t')]$$

or

$$\langle u(\omega)v(-\omega) \rangle = \phi_{uv}(\omega) \quad (\text{A. 11})$$

Using Fourier transform techniques on Eqs. A.1-A.2, we find

$$u(\omega) = n_u(\omega)/(+i\omega+a) \quad ; \quad v(-\omega) = n_v(-\omega)/(-i\omega+b) \quad (\text{A. 12})$$

and thus using Eq. A.11,

$$\phi_{uv}(\omega) = \phi_{nuv}(\omega)/[\omega^2 - i\omega(a-b) + ab]. \quad (\text{A. 13})$$

Integration in complex space leads to (A.8) if we close at $\tau > 0$ around pole (ia) and at $\tau < 0$ around pole $(-ib)$.

These techniques can be generalized to more complicated linear systems:

$$(dx_1/dt) + a_{11}x_1 + a_{12}x_2 = n_1 \quad (\text{A. 14})$$

$$(dx_2/dt) + a_{21}x_1 + a_{22}x_2 = n_2. \quad (\text{A. 15})$$

In time space we reduce these equations to equations of type (A.1-A.2), as was explained in Section IV. In frequency space we immediately apply Fourier-transform techniques to Eqs. A.14-A.15.

We finally mention that Eqs. A.1-A.2 can also be solved for the moments if we first replace them by integral equations.

Note: If, for example, $a = 0$, then the solution for $\phi_{uu}(\tau)$ is not unique: it is determined apart from the constant $\phi_{uu}(0)$. In Section V we encountered that difficulty for ϕ_{BB} . Although the actual value of $\phi_{BB}(0)$ was not important, we have nevertheless mentioned $\phi_{BB}(0) = -1$ if the steady-state density matrix of the field has a Glauber- $p(a)$ representation. From Eqs. 156 and 162 it follows that

$$B = a \exp(i\theta) - a^+ \exp(-i\theta),$$

$$B^2 = a^2 \exp(2i\theta) + (a^+)^2 \exp(-2i\theta) - (aa^+ + a^+a). \quad (\text{A. 16})$$

Therefore, with $a = |a| \exp(-i\theta)$,

$$\phi_{BB}(0) = \int p(a) [|a|^2 + |a|^2 - 2|a|^2 - 1] d^2a. \quad (\text{A. 17})$$

APPENDIX B

Linearization Approximation

The results of Section V for G_1 and G_2 contained quantum and saturation corrections to the predictions of the S. L. theory. In both the S. L. theory and our theory one linearizes the Van der Pol equations (154-155) and consequently makes "linearization errors." We want now to investigate the operating conditions under which our quantum and saturation corrections are more important than the nonlinear corrections of the semiclassical theory.

If we put the substitutions (177) into the semiclassical Van der Pol equations, we find

$$(R_o + R_1)(d\theta/dt) = n'_s \quad (B. 1)$$

$$(dR_1/dt) + 2a\gamma R_o^2 R_1 + 3a\gamma R_o R_1^2 + a\gamma R_1^3 = n'_c = x, \quad (B. 2)$$

where R_o is given by Eq. 159, and n'_s and n'_c are independent Gaussian processes with correlation functions equal to $N_s \delta(\tau) = A\delta(\tau)$. We set

$$\begin{aligned} a &= 2(\gamma - \mu) = 2a\gamma R_o^2 \\ b &= \frac{1}{2} R_o. \end{aligned} \quad (B. 3)$$

In the S. L. theory one approximates Eqs. B. 1-B. 2 by

$$R_o(d\theta/dt) = n'_s \quad (B. 3)$$

$$(dR_1/dt) + aR_1 = n'_c = x, \quad (B. 4)$$

with the result that R_1 and θ are independent Gaussian processes with

$$\langle [\theta(\tau) - \theta(0)]^2 \rangle = A |\tau| / R_o^2 \quad (B. 6)$$

$$\langle R_1(\tau) R_1(0) \rangle = (A/2a) e^{-a|\tau|}. \quad (B. 7)$$

We set

$$A/2a = \overline{R_1^2}. \quad (B. 8)$$

Equation B. 1 is two-dimensional. We can evaluate the first-order influence of R_1 as follows:

$$\theta(\tau) - \theta(0) = \int_0^\tau [R_o + R_1(t')]^{-1} n'_s(t') dt'. \quad (B. 9)$$

We then set in the right-hand side of Eq. B. 9

$$[R_0 + R_1]^{-1} \rightarrow R_0^{-1} [1 - (R_1/R_0)], \quad (\text{B. 10})$$

and use the properties of R_1 as given by the S. L. theory (Gaussian, independent of n'_s). We find immediately from Eq. B. 9

$$\langle [\theta(\tau) - \theta(0)]^2 \rangle = \frac{A|\tau|}{R_0^2} \left[1 - \left(\overline{R_1^2}/R_0^2 \right) \right], \quad (\text{B. 11})$$

and conclude that the nonlinear corrections to the phase diffusion time constant is of order $\left(\overline{R_1^2}/R_0^2 \right)$ of the result of the S. L. theory.

Equation B. 2 is a one-dimensional nonlinear equation that can be solved with the Volterra-kernel method.²² We set

$$R_1 = y_1 + y_2 + y_3 + \dots = A_1 x + A_2 x^2 + A_3 x^3 + \dots \quad (\text{B. 12})$$

and find

$$y_n(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} k_n(\tau_1 \dots \tau_n) x(t-\tau_1) \dots x(t-\tau_n) d\tau_1 \dots d\tau_n, \quad (\text{B. 13})$$

where the first three (symmetric) kernels are given by

$$\begin{aligned} k_1(\tau_1) &= e^{-a\tau_1} \\ k_2(\tau_1, \tau_2) &= 3b \left[e^{-a(\tau_1 + \tau_2)} - e^{-a\tau(1)} \right] \\ k_3(\tau_1, \tau_2, \tau_3) &= b^2 \left[10 e^{-a(\tau_1 + \tau_2 + \tau_3)} + 2 e^{-a[\tau(1) + \tau(2) - \tau(3)]} \right. \\ &\quad \left. - 12 e^{-a[\tau(1) + \tau(2)]} - 6 e^{-a[\tau(1) + \tau(3)]} + 6 e^{-a\tau(1)} \right], \end{aligned} \quad (\text{B. 14})$$

with $\tau(1) > \tau(2) > \tau(3) > 0$. From Eqs. B. 13-B-14 we then find

$$\langle R_1(0) \rangle = -3 \left(\overline{R_1^2}/R_0 \right) \quad (\text{B. 15})$$

$$\langle R_1(\tau) R_1(0) \rangle = \overline{R_1^2} \left\{ e^{-a|\tau|} + \left(\overline{R_1^2}/R_0^2 \right) \left[\frac{9}{4} + e^{-a|\tau|} (3 + 12a|\tau|) + 6 e^{-2a|\tau|} \right] + 0 \left(\overline{R_1^2}/R_0^2 \right)^2 \right\} \quad (\text{B. 16})$$

$$\langle R_1(\tau) R_1(0) R_1(0) \rangle = - \left[\left(\overline{R_1^2} \right)^2 / R_0 \right] \left[3 + (15/2) e^{-a|\tau|} - 3 e^{-2a|\tau|} \right]. \quad (\text{B. 17})$$

We conclude from (B. 16) that the nonlinear corrections to the second-order moment of R_1 are of order $\left(\overline{R_1^2}/R_0^2 \right)$ of the result of the S. L. theory; we see that the first- and

third-order moments are not zero but are of order $\overline{R_1^2}/R_o$, $(\overline{R_1^2}/R_o)^2 R_o$.

If we take into account the way in which these moments appear in the expression of G_2 , we find

(i) The first-order moment gives a DC correction to G_2 which is of order of $\overline{R_1^2}$; this is the basic reason why DC terms in Section V are theoretically meaningless (fortunately they are experimentally unimportant).

(ii) The nonlinear corrections to the second-order moment and the third-order moment give a contribution to G_2 of order $(\overline{R_1^2})^2$, that is, of the same order as the fourth-order moment of R_1 as given by the S.L. theory. (This is the reason why we neglected that fourth-order moment.)

We can now compare the order of magnitude of the nonlinear corrections $\left[\Delta_{n1} = (\overline{R_1^2})^2 / R_o^2 \right]$ with the quantum and saturation corrections ($\Delta_{qs} = 1$) on the relative photon-number fluctuations. We use the numbers of the Freed and Haus experiments.⁴ Under conditions of their measurement closest to threshold, we find for R_o^2 , $\overline{R_1^2}$, Δ_{qs} , Δ_{n1} , respectively, 10^6 , 10^3 , 1, and 1. For an operation point 10 times higher above threshold, we find for these variables 10^7 , 10^2 , 1, and 10^{-3} . We conclude, therefore, that over most of the range of the experiment⁴ the quantum and saturation corrections were much more important than the nonlinear corrections.

APPENDIX C

Transformation to Langevin Form

Throughout this report we have considered the interaction of a system with a reservoir. The system and reservoir are coupled at $t = 0$ and launched in independent initial states. We have been able to express the reservoir variables at t in terms of the reservoir variables at $t = 0$ and the system variables at t , so that the equations of motion of an arbitrary system operator could be cast into the form

$$\frac{ds_o(t)}{dt} = s_1(t)[x_1(t) + a_1]. \quad (C. 1)$$

The operator $x_1(t)$ depends explicitly on time but contains only reservoir operators at $t = 0$, so that the properties of its moments can easily be determined from the initial state of the reservoir. These operators $x(t)$ will be called henceforth "reservoir forces." The term containing the c-number a_1 can represent the influence of the coupling of the system variables among each other, or can have been introduced through the elimination of the reservoir forces at t . Equation C. 1 is only a symbolical representation of the equations of motion of the system variables. First, the right-hand side of (C. 1) represents a sum of terms of similar form; the x factors can eventually appear before the s factors; the s factors associated with the a factors need not be the same as those associated with the x factors. Second, for all operators $s_1(t)$ appearing on the right-hand side of (C. 1), similar equations of motion exist, which will be denoted by $ds_1(t)/dt = s_2(t)[x_2(t) + a_2]$; the same holds true for $s_2(t)$, etc. We remind the reader that the equations of motion for the creation, annihilation, and photon-number operator of the field mode in interaction with the loss system have been reduced to the form (C. 1) (see especially section 4. 8). Also, the equations of motion of the transition operators of a many-level system in interaction with its reservoir have been reduced to the form (C. 1) (see section 6. 3). We mentioned in section 6. 4 that the combined system of field mode and many-level systems of the laser, each interacting with its own reservoir, could be reduced to the form (C. 1).

We shall now show that Eq. C. 1 can be transformed to the Langevin form

$$\frac{Ds_o(t)}{Dt} = A_o(t) + F_o(t), \quad (C. 2)$$

where $A_o(t)$ contains only system operators, $\langle F_o(t) \rangle_R = 0$, and Dt will be defined subsequently. We want, then, to investigate the properties of the "drift" term $A_o(t)$ and of the second-order moments of the "Langevin forces" $F_o(t)$.

We put the following set of sufficient conditions on the reservoir forces, $x(t)$, and on the c-numbers, a . (i) All odd-order moments of the reservoir forces are zero; (ii) The second-order moments of the reservoir forces are proportional to a symmetric Dirac delta function with width τ_R , that is,

$$\langle x_i(t)x_j(t') \rangle_R = w_{ij} \delta(t-t') \quad (C. 3)$$

$$\int_{-\tau_R}^0 \delta(t) dt = \int_0^{\tau_R} \delta(t) dt = \frac{1}{2}. \quad (C. 4)$$

The time duration τ_R is thus of order of the correlation time of the reservoir forces. (iii) All $2n^{\text{th}}$ -order moments are of order of the n^{th} power of the second-order moments. (iv) The reservoir forces at two different times, t and t' , are independent of each other for $|t-t'| \geq \tau_R$. (v) The constant c-numbers, w , a , and τ_R , are all sufficiently small so that a time duration Dt can be found such that

$$\langle s(t)x(t) \rangle_R \tau_R \ll (w \text{ or } a) \langle s(t) \rangle_R Dt \ll \langle s(t) \rangle_R. \quad (C. 5)$$

In order to make the notation somewhat simpler, we shall assume that the ratios of the left-hand sides to the right-hand sides of each of the two inequalities in (C. 5) is of the order of the ratio of wDt to 1.

We shall discuss, first, the qualitative meaning of (C. 5). Note that some of the constants a can depend on the coupling of the system variables among each other; the condition (C. 5) shows, then, that this coupling has to be sufficiently weak; it requires, in general, that natural time dependences have been eliminated from Eq. C. 1. Aside from this remark, assumptions (i-v) clearly involve only the reservoir and the strength of its coupling to the system, since the c-numbers τ_R, w and the other constants a depend only on the reservoir and the strength of its coupling to the system. The only approximations that will be made here will involve: (a) neglecting "overlapping" effects, that is, neglecting terms of the form of the left-hand side of the first inequality in (C. 5) with respect to the right-hand side of that inequality; (b) neglecting "higher order" effects, that is, neglecting terms of the form of the left-hand side of the second inequality in (C. 5) with respect to the right-hand side of that inequality. These approximation errors can thus be made arbitrarily small if the reservoir is chosen in such a way that τ_R is sufficiently small and if its coupling to the system is chosen to be sufficiently weak so that w or a are sufficiently small. We consider this statement to be equivalent to saying that the subsequent proofs are complete and consistent or "exact." Note that we shall not calculate $\langle s(t)x(t) \rangle_R$ in this appendix, but it is possible to show that it is smaller or equal to $(w \text{ or } a) \langle s(t) \rangle_R$, so that the inequalities (C. 5) can be replaced, in fact, by

$$\tau_R \ll Dt \ll (w \text{ or } a)^{-1}. \quad (C. 5)$$

This condition on Dt is equivalent to the condition used by Wangness and Block.²³

From Eq. C. 1, it follows through integration and iteration that

$$\begin{aligned}
\frac{s_o(t+Dt) - s_o(t)}{Dt} &= \frac{1}{Dt} \int_t^{t+Dt} dt_1 s_1(t_1) [x(t_1) + a_1] \\
&= \frac{1}{Dt} \left\{ \int_t^{t+Dt} dt_1 s_1(t) [x_1(t_1) + a_1] + \int_t^{t+Dt} dt_1 \int_t^{t_1} dt_2 s_2(t) [x_2(t_2) + a_2] [x_1(t_1) + a_1] \right. \\
&\quad \left. + \sum_{n=3}^{\infty} \int_t^{t+Dt} dt_1 \dots \int_t^{t_{n-1}} dt_n s_n(t) [x_n(t_n) + a_n] \dots [x_1(t_1) + a_1] \right\}, \quad (C. 6)
\end{aligned}$$

where $t + Dt \geq t_1 \geq t_2 \dots \geq t_n$.

THEOREM 1. The reservoir forces $x(t')$ are independent of the system operators $s(t)$ for $t' \geq t + \tau_R$.

Proof. From the integration of Eq. C. 1, it follows that any system operator at t can be expressed in terms of the system operators at time 0 and reservoir forces at times smaller or equal to t . Since system and reservoir are launched in independent states at $t = 0$, so that the reservoir forces, which depend only explicitly on time, are at all times independent of the system operators at $t = 0$; thus the theorem follows immediately from assumption (iv).

THEOREM 2. The equation of motion of a system operator, averaged over the reservoir, can be found from the a -terms in the first-order term of the perturbation expansion (C. 6) and from the xx -terms of the second-order term of this perturbation expansion, that is,

$$\begin{aligned}
\left\langle \frac{Ds_o(t)}{Dt} \right\rangle_R &= a_1 \langle s_1(t) \rangle_R + \langle s_2(t) \rangle_R \frac{1}{Dt} \int_t^{t+Dt} dt_1 \int_t^{t_1} dt_2 \langle x_2(t_2) x_1(t_1) \rangle_R \\
&= a_1 \langle s_1(t) \rangle_R + \langle s_2(t) \rangle_R \frac{1}{2} w_{21}. \quad (C. 7)
\end{aligned}$$

Other contributions can be neglected insofar as assumption (v) is fulfilled.

Proof. Since the reservoir forces that appear in (C. 6) have time arguments that are later or equal to t , these reservoir forces are independent of the system operators that appear in (C. 6) and have time argument t , except in the range $t, t + \tau_R$. The integrations over this interval give rise to a term of the form $\left\langle (Dt)^{-1} [\exp(s(t)x(t)\tau_R) - 1] \right\rangle_R$, which can thus be neglected with respect to the other contributions, which we shall now show to be those given in (C. 7). For the integration over the remaining range we can use the independence property. All odd-order x -terms give no contribution because of assumption (i). The second-order a -terms and all higher than second-order x -, a - and xa - terms give contributions of order $(w \text{ or } a)^2 Dt$ because of assumption (iii). These contributions can thus be neglected according to assumption (v). The remaining terms, that is, the a -term of the first-order term of the perturbation expansion and the xx -term of the second-order

term of the perturbation expansion, give the contribution calculated in (C. 7). The result (C. 7) is thus correct, except for terms of order $(w \text{ or } a)^2 Dt \langle s(t) \rangle_R$ and $\langle s(t)x(t) \rangle_R \tau_R / Dt$. According to the notation adopted after (C. 5), we should thus multiply the right-hand side of (C. 7) by the factor $[1+0(wDt)]$ to have the correct result. Q. E. D.

We set

$$\frac{Ds_o(t)}{Dt} = A_o(t) + F_o(t), \quad (C. 8)$$

where $A_o(t)$ is defined by

$$A_o(t) = a_1 s_1(t) + \frac{1}{2} w_{21} s_2(t), \quad (C. 9)$$

so that

$$F_o(t) = [-A_o(t)] + \left[\frac{1}{Dt} \int_t^{t+Dt} dt_1 s_1(t) x_1(t_1) \right] + \left[\frac{1}{Dt} \int_t^{t+Dt} dt_1 s_1(t) a_1 \right. \\ \left. + \frac{1}{Dt} \sum_{n=2}^{\infty} \int_t^{t+Dt} dt_1 \dots \int_t^{t_{n-1}} dt_n s_n(t) [x_n(t_n) + a_n] \dots [x_1(t_1) + a_1] \right] \quad (C. 10)$$

$$\langle F_o(t) \rangle_R = 0. \quad (C. 11)$$

Note that we have just proved that (C. 11) has, in fact, a correction term of order $w^2 \langle s(t) \rangle Dt$, because of overlapping and higher order effects. If we neglect that correction term, then the averaged equation of motion becomes a first-order differential equation for the system, that is, it becomes Markovian for first-order moments. In order to simplify notation, we shall denote the three brackets in (C. 10) by 0, 1, and 2, respectively.

THEOREM 3. The Langevin force $F_o(t')$ at a later time is uncorrelated to any system operator at an earlier time (or to any function C of that system operator), that is,

$$\langle C[s(t)] F_o(t') \rangle_R = 0 \quad \text{if } t' \geq t. \quad (C. 12)$$

Proof. If we neglect first overlapping effects, then all reservoir forces in $F_o(t')$ are independent of the system operators in $C[s(t)] F_o(t')$, so that we can average independently over these reservoir forces. This averaging in $[1(t')]$ gives rise to a zero coefficient, and this averaging in $[2(t')]$, according to Theorem 2, gives rise to coefficients that cancel to order $w^2 Dt$ those present in $[0(t')]$. This proves Theorem 3. It is easy to show that overlapping effects and higher-order effects give rise to a correction term of order $0(w^2 Dt \langle C[s(t)] s(t') \rangle)$.

THEOREM 4. The second-order moments of the Langevin forces are proportional to Dirac delta functions with width Dt , and the proportionality constants can be found

from cross-multiplying the x-terms in the first-order term of the perturbation expansion (C. 6) and treating the system operators that appear in these terms as independent of the reservoir forces, that is,

$$\begin{aligned}\langle F_o(t)F_o'(t') \rangle_R &= \langle s_1(t)s_1'(t) \rangle_R \left[\frac{1}{Dt} \int_t^{t+Dt} dt_1 \int_t^{t+Dt} dt'_1 \langle x_1(t_1)x_1'(t'_1) \rangle_R \right] D(t-t') \\ &= \langle s_1(t)s_1'(t) \rangle_R w_{11} D(t-t').\end{aligned}\quad (C.13)$$

$D(t-t')$ is described in Fig. 12. Other contributions may be neglected insofar as assumption (v) is fulfilled.

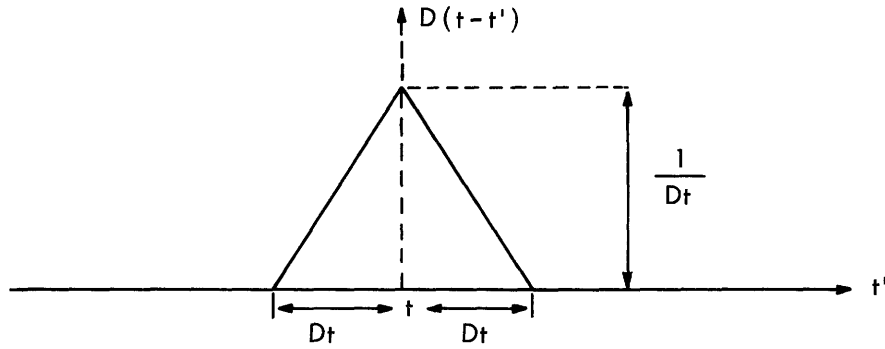


Fig. 12. Dirac delta function describing second-order moments of Langevin forces.

Proof. We consider first the case $t' - t \geq Dt$. We have

$$\langle F_o(t)F_o'(t') \rangle_R = \langle F_o(t)(0'+2') \rangle_R + \langle F_o(t)(1') \rangle_R.$$

If we first neglect overlapping effects, then all reservoir forces in $F_o'(t')$ are independent of all system operators and of all other reservoir forces in $F_o(t)$, so that we can average independently the reservoir forces in $F_o'(t')$. This averaging in (1') gives rise to a zero coefficient, and this averaging in (2') gives rise to coefficients that cancel those in (0') to order w^2Dt . The result (C. 13) for $t' - t \geq Dt$ is thus correct to order

$$\langle F_o(t)s_1'(t')x_1'(t') \rangle_R (\tau_R/Dt) + \langle F_o(t)s'(t')w^2Dt \rangle_R \approx 0 \left(w^2Dt \langle ws(t)s'(t') \rangle_R \right).$$

It is hard to evaluate the left-hand side of this expression exactly because $s'(t')$ is not independent of the reservoir forces present in $F_o(t)$, so that we can no longer average independently over these reservoir forces. In order to obtain an estimate of the magnitude of this left-hand side, we have neglected this correlation effect, but, in turn, have no longer counted on the exact cancellation of (0) and (2) in $F_o(t)$. We obtain, then, the expression at the right-hand side of the expression above, after having used the

notation adopted after (C. 5).

We consider, second, the case $t' = t$. If we neglect the overlapping effects, we can again average independently over the reservoir forces, but now the reservoir forces in $F_o(t)$ and $F'_o(t)$ are not independent of each other, so that this averaging does not give rise to coefficients that all cancel. We have

$$\begin{aligned}\langle F_o(t)F'_o(t) \rangle_R &= \langle (0+1+2)(0'+1'+2') \rangle_R \\ &= \langle (1)(1')+(1)(2')+(2)(1'+2')-(0)(0') \rangle_R.\end{aligned}$$

The first term is given by (C. 13) for $t = t'$. The other terms and the overlapping effects can easily be shown to be of order $w^2 \langle ss' \rangle_R$.

We consider, third, the case in which t' lies in the range $t, t + Dt$. We first expand the system operators at t' in terms of the system operator at t by means of the perturbation expansion (C. 6). It is straightforward to show that the main contribution is given by

$$\langle s_1(t)s'_1(t') \rangle_R \left[\frac{1}{Dt^2} \int_t^{t+Dt} dt_1 \int_{t'}^{t+Dt} dt'_1 \langle x_1(t_1)x'_1(t'_1) \rangle_R \right],$$

which is calculated in (C. 13) for t' in the range $t, t + Dt$. Other contributions are of order $w^2 \langle ss' \rangle$.

We can finally consider the same three cases with $t \geq t'$, and we obtain the same results.

Thus we have the following result

$$\begin{aligned}\langle F_o(t)F'_o(t') \rangle_R &= (C. 13) + 0(w^2 \langle s(t)s'(t) \rangle_R), \quad \text{for } |t-t'| \leq Dt \\ &= (C. 13) + 0(w^2 Dt \langle ws(t)s'(t') \rangle_R), \quad \text{for } |t-t'| \geq Dt.\end{aligned}$$

If we integrate this result over t' [noting that $\int \langle ws(t)s'(t') \rangle_R dt' \approx \langle s(t)s'(t) \rangle_R$], we see that both correction terms relate to the result (C. 13) in the ratio wDt to 1. These correction terms can thus be neglected according to assumption (v).

C. 1 CONCLUSIONS

In sections 3.2 and 6.3, we eliminated reservoir variables at t in terms of reservoir variables at $t = 0$ and system variables at t . We obtained equations of motion of the type (C. 1). Since the reservoir forces then depend only explicitly on time, we can easily verify the fact that the reservoir forces of sections 3.2 and 6.3 obey assumptions (i) to (iv). If we choose Dt so that the inequalities in (C. 5) have the same order-of-magnitude meaning, then we have obtained here results that are correct in the ratio wDt to 1. If we use the form (C. 5) of assumption (v), this ratio is $(\tau_R w)^{1/2}$ to 1. The errors are caused by "overlapping" and "higher order" effects. If this ratio is small, we have

shown that (i) the equation of motion for a first-order system moment is Markovian; (ii) the second-order moments of the Langevin forces are proportional to a Dirac function with width Δt , and Langevin forces at later times are uncorrelated to system operators at earlier times (Lax¹⁰ shows by means of his moment methods that these properties are equivalent to the Markovian character of second-order system moments); (iii) the average equation of motion and the second-order moments of the Langevin forces can be obtained from a second-order perturbation expansion in which one treats the system operators as independent of the reservoir forces. The results of this Appendix, especially Theorems 2 and 4 can now be applied to specific cases. This was done in sections 4.8 and 6.3.

Lax¹⁰ showed that if one assumed property (ii), one can derive a fluctuation-dissipation theorem without approximations. Since we have proved this property (ii), we must necessarily obtain the same fluctuation-dissipation theorem as Lax. (This theorem was discussed in section 6.3.)

We mention, finally, that the assumption (ii) can be modified, that is, we can drop the requirement of symmetry of the Dirac delta functions of Eqs. C.3 and C.4, and work instead with the u -functions defined in Eqs. 33. We see immediately from the first expressions for (C.7) and (C.13) that the average equation will contain frequency shifts (complex coefficients), but the second-order moments of the Langevin forces will only contain real coefficients.

APPENDIX D

Loaded LC Circuit

Throughout this report equivalent classical circuits based on a loaded LC circuit have been discussed. Since we always worked in a normalization in which energy was

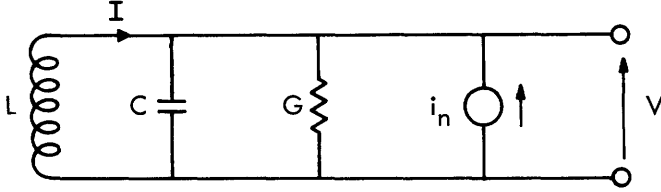


Fig. 13. Loaded LC circuit driven by noise current.

expressed in units of $\hbar\omega_0$, it is useful to repeat these derivations without this normalization.

The loaded LC circuit of Fig. 13 is described by the equations

$$L \frac{dI}{dt} = -V \quad (D. 1)$$

$$C \frac{dV}{dt} + GV - I = i_n. \quad (D. 2)$$

We set

$$\omega_0 = (LC)^{-1/2} \quad (D. 3)$$

$$V = (\hbar\omega_0/2C)^{1/2} i(a-a^\dagger) \quad ; \quad I = (\hbar\omega_0/2L)^{1/2} (a+a^\dagger). \quad (D. 4)$$

Equations D. 3 and D. 4 imply that the energy in the LC circuit is given by

$$\frac{1}{2} LI^2 + \frac{1}{2} CV^2 = \hbar\omega_0 a^\dagger a. \quad (D. 5)$$

If we substitute Eqs. D. 4 in Eqs. D. 1 and D. 2, we find

$$\frac{da}{dt} = -i\omega_0 a - (G/2C)(a-a^\dagger) - i(2\hbar\omega_0 C)^{-1/2} i_n \quad (D. 6)$$

$$\frac{da^\dagger}{dt} = i\omega_0 a^\dagger - (G/2C)(a^\dagger - a) + i(2\hbar\omega_0 C)^{-1/2} i_n. \quad (D. 7)$$

We set

$$i_n = i_n^+(t) e^{i\omega_0 t} + i_n^-(t) e^{-i\omega_0 t} \quad (D. 8)$$

$$a = a(t) e^{-i\omega_0 t} \quad ; \quad a^\dagger(t) e^{i\omega_0 t}. \quad (D. 9)$$

If we now assume that the loaded LC circuit has a high Q and $i_n^+(t)$ and $i_n^-(t)$ contain

only time variations that vary slowly compared with $\exp(\pm i\omega_0 t)$, then we can neglect the double frequency drives in (D. 6) and (D. 7). We obtain for the equations of motion for the new slowly time-variant variables

$$\frac{da(t)}{dt} + ga(t) = x^-(t) \quad ; \quad \frac{da^+(t)}{dt} + ga^+(t) = x^+(t), \quad (D. 10)$$

where

$$g = (G/2C) \quad ;$$

$$x^-(t) = -i(2\hbar\omega_0 C)^{-1/2} i_n^-(t) \quad ; \quad x^+(t) = i(2\hbar\omega_0 C)^{-1/2} i_n^+(t). \quad (D. 11)$$

We shall eventually use time-domain and frequency-domain notation. Spectra and correlations functions are related by

$$\langle u(t+\tau)v(t) \rangle = \int_{-\infty}^{+\infty} \langle u(\omega)v(-\omega) \rangle e^{i\omega\tau} \frac{d\omega}{2\pi},$$

$$\langle u(\omega)v(-\omega) \rangle = \int_{-\infty}^{+\infty} \langle u(t+\tau)v(t) \rangle e^{-i\omega\tau} d\tau. \quad (D. 12)$$

If spectra are said to be "frequency-independent," this will mean that they are frequency-independent over a range around $\omega = 0$, that is, very large compared with the various relaxation constants, but very small compared with ω_0 . We shall then denote these spectra by $\langle uv \rangle$. The variables in both domains can be converted to each other by

$$u(t) \rightarrow u(\omega) \quad ; \quad \frac{du(t)}{dt} \rightarrow i\omega u(\omega). \quad (D. 13)$$

We shall now summarize the various cases in which this equivalent circuit has been applied.

1. Laser Oscillator below Threshold: Model 1. In sections 4. 7 and 4. 8 it was shown that the Glauber functions (that is, moments of normally ordered products of the creation and annihilation operators of the field mode) can be derived from the equivalent loaded LC circuit. In this case, g consists of a loss conductance and gain conductance, and with each of these are associated independent Gaussian noise sources with zero mean. Thus

$$g = \mu - \gamma,$$

$$x^-(t) = x_L^-(t) + x_m^-(t); \quad \text{and complex conjugate.} \quad (D. 14)$$

The only second-order moments different from zero have "frequency-independent" spectra

$$\begin{aligned}\langle x_L^+ x_L^- \rangle &= \langle x_L^- x_L^+ \rangle = 2\mu\beta_L, \\ \langle x_m^+ x_m^- \rangle &= \langle x_m^- x_m^+ \rangle = 2\gamma(1+\beta_m).\end{aligned}\quad (D.15)$$

2. Laser Oscillator below Threshold: Model 2. In section 6.5.1 it was shown that the Glauber functions could be derived from the loaded LC circuit. In this case, g consists of a loss conductance and a frequency-dependent gain conductance. With each of these are associated independent Gaussian noise sources with zero mean. In the frequency domain, we have

$$\begin{aligned}g(\omega) &= \mu - \frac{\Gamma}{i\omega + \Gamma} \gamma \\ x^- (\omega) &= x_L^- (\omega) + \frac{\Gamma}{i\omega + \Gamma} x_m^- (\omega) \quad ; \quad \text{and complex conjugate,}\end{aligned}\quad (D.16)$$

where 2Γ is the bandwidth of material, and x_L^\pm , x_m^\pm are given by (D.15). Note that spectrum of the noise source associated with the frequency-dependent gain conductance is given by

$$\left\langle \frac{\Gamma}{i\omega + \Gamma} x_m^+ (\omega) \frac{\Gamma}{-i\omega + \Gamma} x_m^- (-\omega) \right\rangle = \frac{\Gamma^2}{\omega^2 + \Gamma^2} 2\gamma(1+\beta_m). \quad (D.17)$$

Note also that (D.10) can now be written in time-space as

$$\frac{d^2 a(t)}{dt^2} + (\Gamma + \mu) \frac{da(t)}{dt} + \Gamma(\mu - \gamma) a = \left(\frac{d}{dt} + \Gamma \right) x_L^-(t) + \Gamma x_m^-(t); \quad \text{and complex conjugate} \quad (D.18)$$

3. Laser Oscillator above Threshold: Model 1. The equivalent circuit for this case was discussed in section 5.5. We set

$$g = \mu - \gamma + \alpha \gamma a^+ a \quad ; \quad x^-(t) = x_L^-(t) + x_m^-(t); \quad \text{and complex conjugate.} \quad (D.19)$$

Equations D.1 then become nonlinear. Sufficiently above threshold one can linearize by setting

$$a(t) = [R_s + R_1(t)] e^{-i\theta(t)} \quad ; \quad a^+(t) = [R_s + R_1(t)] e^{i\theta(t)}, \quad (D.20)$$

where R_s is given by the semiclassical steady-state condition

$$\gamma - \alpha \gamma R_s^2 = \mu. \quad (D.21)$$

In the S. L. theory developed by Haus³ one assumes

$$\langle x_m^+ x_m^- \rangle = 2\mu(1+\beta_m^s) \quad ; \quad \langle x_L^+ x_L^- \rangle = 2\mu\beta_L. \quad (D.22)$$

One could just as well have set

$$\langle x_m^+ x_m^- \rangle = 2\mu \left(\frac{1}{2} + \beta_m^s \right) = 2\gamma \left(\frac{1}{2} + \beta_m \right) \quad ; \quad \langle x_L^+ x_L^- \rangle = 2\mu \left(\frac{1}{2} + \beta_L \right). \quad (\text{D. 23})$$

For the noise sources,

$$n_s = (1/2i)(x^- e^{i\theta} - x^+ e^{-i\theta}) \quad ; \quad n_c = (1/2)(x^- e^{i\theta} + x^+ e^{-i\theta}), \quad (\text{D. 24})$$

which are the important noise sources that enter into the solution for the phase noise and amplitude noise. We obtain

$$\begin{aligned} \langle n_s n_s \rangle &= \langle n_c n_c \rangle = N_s \quad ; \quad \langle n_c n_s \rangle = 0 \quad ; \\ N_s &= \gamma \left(\frac{1}{2} + \beta_m \right) + \mu \left(\frac{1}{2} + \beta_L \right) = \mu \left(\frac{1}{2} + \beta_m^s + \frac{1}{2} + \beta_L \right) = \mu (1 + \beta_m^s + \beta_L). \end{aligned} \quad (\text{D. 25})$$

If we want to rederive the Glauber functions, including the quantum and saturation corrections, we have to set

$$\begin{aligned} \langle n_s n_s \rangle &= N_s \quad ; \quad \langle n_c n_s \rangle = 0 \\ \langle n_c n_c \rangle &= \mu \left(\frac{1}{2} + \beta_m^s \right) \left(1 - 4 \frac{\gamma - \mu}{\mu} \right) + \mu \left(\frac{1}{2} + \beta_L \right) - (\gamma - \mu). \end{aligned} \quad (\text{D. 26})$$

If we want to rederive the correlation function of the photon number operator, we drop the last term of (D. 26). This is then equivalent to

$$\begin{aligned} \langle x_m^+ x_m^- \rangle &= 2(\gamma - \alpha \gamma R_s^2) \left(\frac{1}{2} + \beta_m \right) = 2\mu \left(\frac{1}{2} + \beta_m^s \right) (\mu/\gamma), \\ \langle (x_m^+ e^{-i\theta})^2 \rangle &= \langle (x_m^- e^{i\theta})^2 \rangle = -2\alpha \gamma R_s^2 \left(\frac{1}{2} + \beta_m \right) = -2(\gamma - \mu) \left(\frac{1}{2} + \beta_m^s \right) (\mu/\gamma). \end{aligned} \quad (\text{D. 27})$$

The importance of these correction terms has been discussed in Section V. We shall refer to them in Case 4.

4. Laser Oscillator above Threshold. This case was discussed in Section VI. We showed how to derive the phase fluctuations and the correlation function of the photon-number operator. The quantum corrections on the second-order Glauber functions were only discussed qualitatively. Apart from these quantum corrections, we can rederive these correlation functions from the loaded LC circuit, provided we set

$$g(\omega) = \mu - (\kappa^2/\Gamma) P(t) \frac{\Gamma}{i\omega + \Gamma} \quad ; \quad x^-(\omega) = x_L^-(\omega) + x_m^-(\omega) \frac{\Gamma}{i\omega + \Gamma} \quad (\text{D. 28})$$

$$\langle x_L^+ x_L^- \rangle = 2\mu \left(\frac{1}{2} + \beta_L \right) \quad ; \quad \langle x_m^+ x_m^- \rangle = 2\mu \left(\frac{1}{2} + \beta_m^s \right),$$

so that Eqs. D. 10 can be written in time-space

$$\frac{d^2 a(t)}{dt^2} + (\Gamma + \mu) \frac{da(t)}{dt} + \Gamma \left[\mu - \frac{\kappa^2}{\Gamma} P(t) \right] a(t) = \left(\frac{d}{dt} + \Gamma \right) x_L^-(t) + \Gamma x_m^-(t). \quad (\text{D. 29})$$

The variable $P(t)$ obeys a set of rate equations described by Eq. 222 and Eqs. 223c, with all variables now c-numbers. We linearize all of these equations by setting

$$a(t) = [R_s + R_1(t)] e^{-i\theta(t)} \quad ; \quad a^\dagger(t) = [R_s + R_1(t)] e^{i\theta(t)};$$

$$P = P_s + P_1(t) \quad ; \quad Q = Q_s + Q_1(t); \quad \text{etc.}, \quad (\text{D. 30})$$

where

$$\mu = (\kappa^2/\Gamma)P_s \quad ; \quad 4\mu R_s^2 = -\Gamma_{pp}P_s + \Gamma_{pq}Q_s + R_P^s; \quad (\text{D. 31})$$

and obtain Eqs. 260, with all variables now c-numbers, so that, in particular, all noise sources commute. The noise source x^\pm should be understood as $x_L^\pm + x_m^\pm$ in these equations. All noise sources are Gaussian and have zero mean. The noise sources X_P , X_Q , etc. are shot-noise sources; their second-order moments are given in Eqs. 225. They are independent of x_L^\pm but not of x_m^\pm . In particular,

$$\left\langle \left[x_m^- e^{i\theta} + x_m^+ e^{-i\theta}, X_P \right]_+ \right\rangle = -4\mu R_s (\Gamma_{pq}/\Gamma). \quad (\text{D. 32})$$

Similar anticommutators for X_Q , X_{kk} can easily be found from Eqs. 225. We finally mention that if one calculates moments involving the noise sources of the type $x^+ a + x^- a^\dagger$ or $x^+ e^{-i\theta} + x^- e^{i\theta}$, one should consider a^\pm and $\exp(\pm i\theta)$ as independent of x^\pm .

The solution of this set of coupled equations can become very involved. We have only solved them in Section VI for the special case in which the bandwidth of the intensity fluctuations is small compared with $(\Gamma + \mu)$, (so that the equations can be transformed to "modified" rate equations) and the material system consists of two-level systems or of highly inverted systems. McCumber has used the additional condition $\epsilon = \Gamma/(\Gamma + \mu) = 1$ and neglected the correlation between X_+ and x_m^\pm . We have mentioned in Section VI that in the limiting case

$$\frac{1}{2} \Gamma_{pp} = \Gamma = T^{-1} \gg \mu$$

Case 4 reduces to Case 3, except, of course, for the quantum corrections. The saturation corrections of Case 3 are introduced in Case 4 through the influence of noise sources in the equations for P , Q , etc., and especially through the correlation of these noise sources with the noise sources x_L^\pm and x_m^\pm .

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